A Lelek-like compact metrizable space

Joint ongoing work with R. Camerlo

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Overview

1. Approximating compact metrizable $\mathcal{L}$-structures
2. An introduction to projective Fraïssé theory
3. A universal space and its characterization
4. Open problems

Approximating compact metrizable $\mathcal{L}$-structures

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Suppose $\mathcal{L} = (S_i)_{i \in \omega}$ is a a countable relational language, and the arity of S_i is s_i.

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In a recent paper[1], the authors proposed the following definition.

Suppose $\mathcal{L} = (S_i)_{i\in\omega}$ is a a countable relational language, and the arity of S_i is s_i. A compact metrizable $\mathcal{L}$-structure is a tuple $(X, (S_i^X)_{i\in\omega})$, where X is a compact metrizable space and $S_i^X \subseteq X^{s_i}$ is closed, for each i.

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2. $\overline{Y \setminus H} = Y'$,
3. $H = \bigcap_{n\in\omega} \bigcup \mathcal{U}_n$,

where $\mathcal{U}_n$ is a finite collection of pairwise disjoint open subsets of Y and for each n, $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$, that is, for each U in $\mathcal{U}_{n+1}$ there is $U' \in \mathcal{U}_n$ such that $U \subseteq U'$.

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Notice that condition 2 implies that the maximum of the diameters of the sets in $\mathcal{U}_n$ goes to zero as n grows.

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Approximate a space with a projective sequence of open covers

Define:

$$\mathcal{U}_\infty = \left\{ (U_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{U}_n \;\middle|\; U_{n+1} \subseteq U_n \right\}$$

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Then we give $\mathcal{U}_\infty$ an $\mathcal{L}_R$ structure by letting:

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- $((U_n)_{n\in\omega}, (U'_n)_{n\in\omega}) \in R^{\mathcal{U}_\infty}$ if and only if, for each $n \in \omega$, $(U_n, U'_n) \in R^{\mathcal{U}_n}$.

So $\mathcal{U}_\infty$ is a compact metrizable $\mathcal{L}_R$-structure.

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Proof. Let $q_Y : \mathcal{U}_\infty \to Y$ be $q((U_n)_{n\in\omega}) = \bigcap_{n\in\omega} \overline{U_n}$.

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Let G, G' be compact metrizable $\mathcal{L}_R$-structures. An epimorphism $\phi : G' \to G$ is a continuous surjective function such that:

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So, given a sequence $G_0 \xleftarrow{\phi_0} G_1 \xleftarrow{\phi_1} G_2 \cdots$, we can define the projective limit of (G_n, ϕ_n) as

$$G_\infty = \left\{ (a_n)_{n\in\omega} \in \prod_{n\in\omega} G_n \;\middle|\; \forall n, \phi_n(a_{n+1}) = a_n \right\}.$$

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$$d_R\left(\phi_{m-1}^{-1} \cdots \phi_n^{-1}(a), \phi_{m-1}^{-1} \cdots \phi_n^{-1}(a')\right) \geq 3.$$

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A sequence (G_n, ϕ_n) is fine if and only if R^{G_∞} is an equivalence relation. Say that (G_n, ϕ_n) approximates G_∞ / R^{G_∞}.

An introduction to projective Fraïssé theory

Topological → combinatorial

[2] Trevor Irwin and Sławomir Solecki (2006). "Projective Fraïssé limits and the pseudo-arc". In: Trans. Amer. Math. Soc.

Topological $\to$ combinatorial

Given a class $\mathcal{C}$ of compact metrizable $\mathcal{L}$-structures we can look at a class Γ of finite $\mathcal{L}_R$-structures such that each $Y \in \mathcal{C}$ is approximated by a fine sequence of Γ.

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A compact metrizable space ($\mathcal{L} = \emptyset$) is connected if and only if it can be approximated by a fine sequence of connected R-graphs.

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Theorem (Irwin-Solecki, 2006[2])

A compact metrizable space ($\mathcal{L} = \emptyset$) is chainable and connected if and only if it can be approximated by a fine sequence of finite connected linear R-graphs.

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$$H_{i_0} \xleftarrow{\hat{\chi}_0} H_{i_1} \xleftarrow{\hat{\chi}_1} H_{i_2} \cdots ,$$

where $\hat{\chi}_n = \chi_{i_n}\chi_{i_n+1}\cdots\chi_{i_{n+1}-1}$, and epimorphisms $f_n : H_{i_n} \to G_n$ such that $\phi_n f_{n+1} = f_n \hat{\chi}_n$.

$$\begin{array}{ccccccc}
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\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\
G_0 & \xleftarrow[\phi_0]{} & G_1 & \xleftarrow[\phi_1]{} & G_2 & & \cdots
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If $H_0 \xleftarrow{\chi_0} H_1 \xleftarrow{\chi_1} H_2 \cdots$ is a universal fine sequence for Γ it follows that H_∞/R^{H_∞} is projectively universal for all compact metrizable $\mathcal{L}$-structures approximated by sequences in Γ, since $f_\infty = (f_n)_{n\in\omega}$ induces an epimorphism on the quotients:

$$\begin{aligned} q^*(f_\infty) : X = H_\infty/R^{H_\infty} &\to G_\infty/R^{G_\infty} = Y \\ x &\mapsto q_Y f_\infty q_X^{-1}(x). \end{aligned}$$

Fraïssé Theory

A class Γ of finite compact metrizable $\mathcal{L}_R$-structures such that:

- (JPP) $\forall G, G' \in \Gamma, \exists H \in \Gamma$ and epimorphisms $\phi : H \to G$, $\phi' : H \to G'$;

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Some consequences

Let Γ be a Fraïssé class of finite $\mathcal{L}_R$-structures whose sequences approximate the compact metrizable $\mathcal{L}$-structures of a class $\mathcal{C}$, and $H_0 \xleftarrow{\chi_0} H_1 \xleftarrow{\chi_1} H_2 \cdots$ be a fine universal sequence for Γ. Denote H_∞/R^{H_∞} by $X_{\mathcal{C}}$.

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- **approximate projective homogeneity:** let $Y \in \mathcal{C}$ and $f, f' : X_\mathcal{C} \to Y$ be epimorphisms, then, for any $\epsilon > 0$, there exists an $\mathcal{L}$-isomorphism $\alpha : X_\mathcal{C} \to X_\mathcal{C}$ such that for any $x \in X_\mathcal{C}$, $d(f(x), f'\alpha(x)) < \epsilon$;

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- any $\mathcal{L}$-isomorphism $h : X_\mathcal{C} \to: X_\mathcal{C}$ uniformly approximable by $\mathcal{L}_R$-isomorphisms $\alpha_\infty : H_\infty \to H_\infty$.

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A universal space and its characterization

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Let $\mathcal{L} = \{\leq\}$. A compact metrizable $\mathcal{L}_R$-structure A is a Hasse diagram of a partial order if $\leq^A$ is a partial order and xR^Ax' if and only if $x = x'$ or x is the immediate predecessor or successor of x' wrt $\leq^A$.

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Let $\Pi_\triangledown$ be the class of all Hasse diagram of finite partial orders which do not contain R-cycles.

Theorem (B.- Camerlo)

$\Pi_\triangledown$ is a projective Fraïssé class, whose universal sequence $P_0 \xleftarrow{\chi_0} P_1 \xleftarrow{\chi_1} P_2 \cdots P_\infty$ is fine.

Smooth fences

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A fence is smooth if and only if it can be embedded in the Cantor fence $2^{\mathbb{N}} \times [0,1]$*, preserving the order.*

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Theorem (B.-Camerlo)

Any smooth fence can be approximated by a fine projective sequence of $\tilde{\Pi}_{\nabla}$.

Characterization theorem

An endpoint of a compact metric space Y is a point x such that, for any embedding $h : [0, 1] \to Y$ such that $x \in ran(h)$, $x = h(0)$ or $x = h(1)$.

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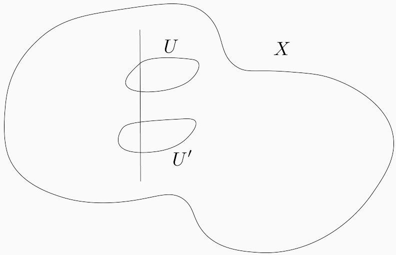

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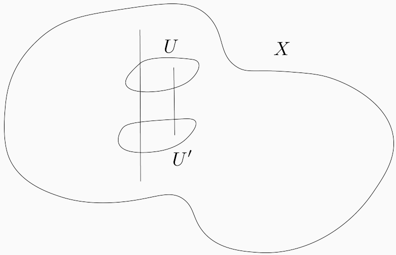

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Let X be a nonempty smooth fence such that for any open sets U, U' which both meet a common connected component of X, there is an arc of X whose endpoints belong to U, U', respectively.

Then X is homeomorphic to P_∞ / R^{P_∞}.

Outline of the proof of the characterization

- Consider a space X which satisfies the assumptions of the theorem.
- Find an appropriate fine projective sequence $X_1 \leftarrow X_2 \cdots$ of Π_∇ which approximates X.
- Prove that such a sequence is a universal sequence for Π_∇.
- Conclude that X_∞ is isomorphic to P_∞ by uniqueness of the projective Fraïssé limit and thus that their quotients are homeomorphic.

Approximate projective homogeneity

Theorem (B.- Camerlo)

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Question: What are larger classes of spaces for which the previous theorem holds? Can we characterize the quotients of projective limits of Π_∇?

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If Y is a compact metric space and $u, v \in Y$, denote by $[u, v]$ the intersection of all closed connected subsets of Y containing both u, v.

A fan is smooth if the partial order $x \preceq y \iff [t, x] \subseteq [t, y]$ is closed. Equivalently if it can be embedded in the Cantor fan $(2^{\mathbb{N}} \times [0, 1])/(x, 0) \sim (x', 0)$.

The Lelek fan is the unique smooth fan whose set of endpoints is dense.

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Theorem (Bartošová-Kwiatkowska, 2015)

The class of all finite partial orders with a minimum and which do not contain R-cycles is a projective Fraïssé class with a fine universal sequence the quotient of whose limit is homeomorphic to the Lelek fan.

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Open problems

Theorem (Bartošová-Kwiatkowska, 2017[3])

The universal minimal flow of the group of homeomorphisms of the Lelek fan is the space of maximal closed chains of the Lelek fan which are downward closed and connected.

[3] Dana Bartošová and Aleksandra Kwiatkowska (2017). “Universal minimal flow of the homeomorphism group of the Lelek fan”. In: ArXiv e-prints. arXiv: 1706.09154 [math.LO].

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Question: What is the universal minimal flow of the group of homeomorphisms of P_∞/R^{P_∞}?

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