Set-Theoretic Methods in Topology and Real Functions Theory

dedicated to 80th birthday of Lev Bukovský

September 9th–13th 2019, Košice, Slovakia

Invited speakers

- Aleksander Błaszczyk
- Vera Fischer
- István Juhász
- Menachem Magidor
- Dilip Raghavan

Winter School 2019

January 26th–February 2nd 2019
Hejnice, Czech Republic

Invited speakers

- James Cummings
- Miroslav Hušek
- Wiesław Kubiś
- Jordi Lopez-Abad

www.winterschool.eu

Logic Colloquium 2019

August 11th–16th 2019, Prague, Czech Republic

www.lc2019.cz

Program Committee

- Andrew Arana
- Lev Beklemishev (chair)
- Agata Ciabattoni
- Russell Miller
- Martin Otto
- Pavel Pudlák
- Stevo Todorčević
- Alex Wilkie

Splitting Chains

David Chodounský

Institute of Mathematics CAS

joint work of Félix Cabello, Antonio Avilés, Piotr Borodulin-Nadzieja, David Chodounský, and Osvaldo Guzmán

Splitting Chains

For $A, S \subset \omega$ we say that S *splits* A if both $A \cap S$ and $A \setminus S$ are infinite. Otherwise A *reaps* S.

$\mathcal{S} \subset \mathcal{P}(\omega)$ is *splitting* if for each infinite $A \subset \omega$ there exists $S \in \mathcal{S}$ such that S splits A.

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Do splitting chains exist?

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Do splitting chains exist?

Answer: Sometimes

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Question

Do splitting chains exist?

Answer: Sometimes, e.g. under CH.

Exact sequences of Banach spaces

Definition

An *exact sequence* of Banach spaces is a diagram

$$0 \longrightarrow Y \stackrel{\imath}{\longrightarrow} Z \stackrel{\pi}{\longrightarrow} X \longrightarrow 0$$

of Banach spaces and linear continuous operators such that the kernel of each arrow agrees with the range of the preceding one. The exact sequence is *trivial* if there is an operator $\varpi\colon Z \to Y$ such that $\varpi \circ \imath = \mathrm{id}_Y$.

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Theorem

Every exact sequence of the form

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False, counterexamples are constructed using splitting chains.

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Theorem

If there is a splitting chain of size κ, then there is a nontrivial exact sequence

$$0 \longrightarrow \ell_\infty/c_0 \xrightarrow{\imath} Z \xrightarrow{\pi} c_0(\kappa) \longrightarrow 0.$$

Tunnels in topological spaces

Let U, V be open subsets of a topological space X.
We write $U < V$ when $\overline{U} \subseteq V$.
A family of open subsets of X is a *chain* if it is linearly ordered by $<$.

Definition (Nyikos)

Chain $\mathcal{U}$ of open subsets of a topological space X is a *tunnel* in X if the set $\bigcup\{\, \overline{U} \setminus U : U \in \mathcal{U} \,\}$ is dense in X.

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Theorem (Marciszewski)

If X is metrizable without isolated points, then there is a tunnel in X.

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Theorem

Stone spaces of Aronszajn algebras do not have tunnels.

Theorem

The Suslin hypothesis is equivalent to the assertion that every c.c.c. compact zero-dimensional space without isolated points has a tunnel.

Tunnels and chains in topological spaces

For A, S open subsets of a topological space we say that S *splits* A if both $A \cap S$ and $A \setminus S$ are non-empty.

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Theorem

Let K be a compact space. The following are equivalent:

1. *K has a tunnel,*
2. *K has a splitting chain of open sets,*
3. *there is a continuous mapping $f: K \to L$ into a linearly ordered space L, and with nowhere dense fibers. I.e. $f^{-1}[\{l\}]$ is nowhere dense for each $l \in L$.*

Pre-gaps in $\mathcal{P}(\omega)$

We say that $(\mathcal{L}, \mathcal{R})$ is a *(linear) pre-gap* if $\mathcal{L}, \mathcal{R} \subset \mathcal{P}(\omega)$, for each $L \in \mathcal{L}, R \in \mathcal{R}$ is $L \subset^* R$, and $\mathcal{L}, \mathcal{R}$ are linearly ordered by $\subset^*$.

$S \subset \omega$ *separates* a pre-gap $(\mathcal{L}, \mathcal{R})$ if $L \subset^* S \subset^* R$ for each $L \in \mathcal{L}, R \in \mathcal{R}$.

$S \subset \omega$ *spreads* a pre-gap $(\mathcal{L}, \mathcal{R})$ if $L \cap S =^* \emptyset$ and $S \subset^* R$ for each $L \in \mathcal{L}, R \in \mathcal{R}$.

A pre-gap is a *gap* if there is no S separating it.

A pre-gap is *tight* if there is no S spreading it.

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Observation

If A, B separate a tight pre-gap, then $A =^* B$.

Let S be a chain in $\mathcal{P}(\omega)$. We say that a pre-gap $(\mathcal{L}, \mathcal{R})$ is a *cut* in S if $S = \mathcal{L} \cup \mathcal{R}$.

Proposition

Chain S in $\mathcal{P}(\omega)$ is splitting iff every cut in S is a tight pre-gap.

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If S is a splitting chain, then $|S| = \mathfrak{c}$.

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Corollary

Let $V \subset W$ be models of ZFC, such that there is a real number in $W \setminus V$. If $S \in V$, then S is not a splitting chain in W.

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An (ω_1, ω_1) gap exists in ZFC.
PFA implies that every gap of type $(\omega_1, ?)$ has type (ω_1, ω_1).

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Theorem

A tight (ω_1, ω_1) pre-gap exists iff $\mathfrak{p} = \omega_1$.

Corollary

PFA implies that there are no splitting chains in $\mathcal{P}(\omega)$.

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Theorem

Assume CH and let κ be of uncountable cofinality. Then $\mathbf{C}_\kappa$ forces that there is splitting chain.

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Let κ be a regular uncountable cardinal. It is consistent that $\mathfrak{c} = \kappa$, MA(σ-centered), and a splitting chain exists.

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Let $\mathcal{G} = (\mathcal{L}, \mathcal{R})$ be a pre-gap. The separating forcing $\mathbf{P}_\mathcal{G}$ consists of conditions $p = (s_p, L_p, R_p)$ such that $s_p \in 2^{<\omega}$, $L_p \in [\mathcal{L}]^{<\omega}$, $R_p \in [\mathcal{R}]^{<\omega}$, and $L \setminus |s_p| \subset R$ for each $L \in L_p$, $R \in R_p$.
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If the type of $\mathcal{G}$ is $(\leq \omega, \leq \omega)$ then $\mathbf{P}_\mathcal{G}$ is the Cohen forcing.

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Let G be a $\mathbf{P}_{\mathcal{G}}$ generic filter. The set $S = \bigcup\{ s_p : p \in G \}$ separates $\mathcal{G}$.

Proposition

If $A \in V$ spreads $\mathcal{G}$, then S splits A.

Proposition

Let $\mathcal{G}$ be a tight gap. Adding any number of Cohen reals does not add a subset spreading $\mathcal{G}$.