Ideal convergent subseries and rearrangements of series in Banach spaces

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results obtained together with Michał Popławski and Artur Wachowicz

We will consider proper ideals of subsets of $\mathbb{N}$, containing Fin, the family of finite sets. An ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ can be treated as a subset of the Cantor space $\{0,1\}^{\mathbb{N}}$ (via a bijection $\mathcal{P}(\mathbb{N}) \sim \{0,1\}^{\mathbb{N}}$.)

Theorem (Talagrand; Jalali-Naini)

An ideal $\mathcal{I}$ on $\mathbb{N}$ has the Baire property
$\Leftrightarrow$ there exists a sequence $n_1 < n_2 < \dots$ of natural indices such that no member of $\mathcal{I}$ contains infinitely many intervals $I_k := [n_k, n_{k+1}) \cap \mathbb{N}$.

Consider the following Polish subspaces of the Polish space $\mathbb{N}^{\mathbb{N}}$:
$S := \{s \in \mathbb{N}^{\mathbb{N}} \colon \forall n \in \mathbb{N}\ s(n) < s(n+1)\}$
$P := \{p \in \mathbb{N}^{\mathbb{N}} \colon p \text{ is a bijection}\}$.
Then S codes subseries of a series, and P codes its rearrangements.

Fact 1 [Rao-Rao-Rao]

If a series $\sum x_n$ is divergent (is not absolutely convergent) in $\mathbb{R}$, then almost all, in the sense of the Baire category, its subseries (rearrangements) are divergent.

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Fact 2 [Dindos, Šalát, Toma]

If a series $\sum x_n$ is stat-divergent in $\mathbb{R}$, then almost every, in the sense of category, its subseries is stat-divergent.

Definition

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. A sequence (a_n) in a normed space X is called $\mathcal{I}$-convergent to $a \in X$, if $\{n \in \mathbb{N} : ||a_n - a|| > \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$. [Fin-convergence is the usual convergence].

Remark: Every $\mathcal{I}$-convergent sequence is $\mathcal{I}$-bounded, i.e. $\{n \in \mathbb{N} : ||a_n|| > M\} \in \mathcal{I}$ for some $M > 0$.

Example

For any set $A \subset \mathbb{N}$, define
$d(A) := \lim_{n\to\infty} \frac{|A \cap \{1,\ldots,n\}|}{n}$.
Then $\mathcal{I}$-convergence generated by the ideal
$\mathcal{I}_d := \{A \subset \mathbb{N} : d(A) = 0\}$ is called statistical convergence.

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The $\mathcal{I}$-convergence of a series means the $\mathcal{I}$-convergence of the sequence of its partial sums.

Our aim was to generalize Facts 1 and 2 as far as possible.

• We consider a series $\sum x_n$ that is not unconditionally convergent in a Banach space.

• We consider an ideal $\mathcal{I}$ on $\mathbb{N}$ with the Baire property.

• We study the Baire category of the sets
$A := \{s \in S \colon \sum x_{s(n)}$ is $\mathcal{I}$-convergent$\}$ and
$B := \{p \in P \colon \sum x_{p(n)}$ is $\mathcal{I}$-convergent$\}$ in S and P, respectively.

Theorem 1 [BPW1]

Assume that a series $\sum x_n$ is not unconditionally convergent in the Banach space X.
Assume that an ideal $\mathcal{I}$ with the Baire property on $\mathbb{N}$ is 1-shift invariant.
Then the sets A and B are meager in S and P, respectively.

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Definition [Leonov]

An ideal $\mathcal{I}$ on $\mathbb{N}$ is called 1-shift invariant, if
$(\forall A \subset \mathbb{N})\ A \in \mathcal{I} \Rightarrow \mathbb{N} \setminus (A+1) \notin \mathcal{I}$.
Remark: This is weaker than the shift-invariance of $\mathcal{I}$:
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Let $\mathcal{I}$ be an ideal $\mathcal{I}$ on $\mathbb{N}$.
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Proof of Theorem 1 (for the set A; sketch).

• We may assume that $\liminf_n ||x_n|| = 0$ since, if $\liminf_n ||x_n|| > 0$, then $A = \emptyset$ and the assertion holds.
Indeed, suppose that there exists $s \in A$. Then $\sum_n x_{s(n)}$ is $\mathcal{I}$-convergent. Since $\mathcal{I}$ is 1-shift invariant, by the Leonov theorem, we have $\liminf_n ||x_{s(n)}|| = 0$. A contradiction.
• We use the characterization due to Orlicz: *A series $\sum y_n$ is unconditionally convergent in a Banach space $\Leftrightarrow$ every subseries $\sum_n y_{s(n)}$, $s \in S$, is convergent.*
In our case, $\sum_n x_n$ is not unconditionally convergent. Hence pick $u \in S$ such that $\sum_n x_{u(n)}$ is divergent, so the Cauchy condition does not hold. That is,

$$\exists \varepsilon > 0 \, \forall m \in \mathbb{N} \, \exists t_m > m \, \left\| \sum_{m+1}^{t_m} x_{u(i)} \right\| > 2\varepsilon.$$

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$\forall \eta > 0\ \exists m \in \mathbb{N}\ \{j > m\colon \| \sum_{i=m+1}^{j} x_{s(i)} \| > \eta\} \in \mathcal{I}$.

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To this aim, we use: the Talagrand characterization,
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$$A_{\eta m} := \left\{ s \in S \colon \left\{ j > m \colon \left\| \sum_{i=m+1}^{j} x_{s(i)} \right\| > \eta \right\} \in \mathcal{I} \right\}.$$

Thus in particular, $A \subset \bigcup_{m \in \mathbb{N}} A_{\varepsilon m}$ where ε is chosen as above.

• Then we show that every set $A_{\varepsilon m}$, $m \in \mathbb{N}$, is meager.
To this aim, we use: the Talagrand characterization,
the divergence of $\sum_n x_{u(n)}$, and the condition $\liminf_n ||x_n|| = 0$.

The proof for B is similar. □

Remarks

• If $dim(X) = \infty$, there exists an unconditionally convergent series in X, which is not absolutely convergent [Dvoretzky-Rogers]. For such a series, we have $A = S$ and $B = P$.
Hence Theorem 1 is not valid where the lack of unconditional convergence is replaced by the lack of absolute convergence.

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For $dim(X) < \infty$, we have a stronger result. It can be stated for $X = \mathbb{R}$.

Theorem 2

Assume that a series $\sum x_n$ is not absolutely convergent in $\mathbb{R}$.
Assume that an ideal $\mathcal{I}$ on $\mathbb{N}$ has the Baire property.
Then the following sets
$E := \{s \in S \colon (\sum_{i=1}^{n} x_{s(i)})_{n\in\mathbb{N}}$ is $\mathcal{I}$-bounded$\}$;
$F := \{p \in P \colon (\sum_{i=1}^{n} x_{p(i)})_{n\in\mathbb{N}}$ is $\mathcal{I}$-bounded$\}$
are meager in S and P, respectively.

Note that $A \subset E$ and $B \subset F$. We do not assume that an ideal $\mathcal{I}$ is 1-shift invariant.

The scheme of the proof of Theorem 2 is similar to that used for Theorem 1. We apply the alternative
$\sum_{x_n>0} x_n = \infty$ or $\sum_{x_n\leqslant 0} = -\infty$
which follows from the assumption $\sum_n |x_n| = \infty$.

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Subseries can be coded in a different way, by the use of the set

$$T := \{t \in \{0,1\}^{\mathbb{N}} \colon t(n) = 1 \text{ for infinitely many } n's\}$$

which is a Polish subspace of $\{0,1\}^{\mathbb{N}}$.
Then $\sum_n t(n)x_n$ for $t = (t(n)) \in T$ is a subseries of a series $\sum_n x_n$.

We observed that for some ideals $\mathcal{I}$ the methods of coding of subseries by the sets S and T produce different classes of $\mathcal{I}$-convergent subseries. [For $\mathcal{I} :=$ Fin they are the same.]

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Under the approach with the set T, the assertion of Theorem 1 for subseries remains true without assumption about 1-shift invariance of $\mathcal{I}$.

Theorem 1' [BPW2]

Assume that a series $\sum_n x_n$ is not unconditionally convergent in the Banach space X.
If $\mathcal{I}$ is an ideal with the Baire property, then the set

$$A^* := \left\{ t \in T\colon \sum_n t(n)x_n \text{ is } \mathcal{I}\text{-convergent} \right\}$$

is meager in T.

Remark: The set T is co-countable in $\{0,1\}^N$, so we can consider the whole space $\{0,1\}^{\mathbb{N}}$ instead of T, treating a series $\sum_n t(n)x_n$ with $t \notin T$ as convergent.

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Measure case

Let $\nu(\{0\}) = 1/2 = \nu(\{1\})$, and consider the product measure λ on $\{0,1\}^{\mathbb{N}}$ generated by ν (the Haar measure on $\{0,1\}^{\mathbb{N}}$).

Given an ideal $\mathcal{I}$ on $\mathbb{N}$, and an $\mathcal{I}$-divergent series $\sum_n x_n$ in a Banach space X, let us consider its subseries. One can ask:

- Is the set $A(\mathcal{I}) := \{t \in \{0,1\}^{\mathbb{N}} \colon \sum_n t(n)x_n \text{ is } \mathcal{I}\text{-convergent}\}$ measurable?
- What is the value $\lambda(A(\mathcal{I}))$?

For $\mathcal{I} :=$ Fin, and $\mathcal{I} := \mathcal{I}_d, X := \mathbb{R}$, we have $\lambda(A(\mathcal{I})) = 0$ [Dindoš, Šalát et als].

We will generalize these results.

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Theorem 3

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ which is analytic or coanalytic. Let $\sum_n x_n$ be a series in a Banach space. Then $\lambda(A(\mathcal{I}))$ is either 0 or 1. If $\sum_n x_n$ is $\mathcal{I}$-divergent, then $\lambda(A(\mathcal{I})) = 0$.

In the proof, we use 0-1 law for measure.

Dindoš, Šalát and Toma proved that for $\mathcal{I} := \mathcal{I}_d$, the second assertion of is not valid if we replace $\mathcal{I}$-divergence of $\sum_n x_n$ by its divergence.

They gave an example of a divergent series with such that $\lambda(A(\text{Fin})) = 0$ while as $\lambda(A(\mathcal{I}_d)) = 1$.

We will use a similar method to obtain the same effect for a wide class of ideals.

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Definition

We say that an ideal $\mathcal{I}$ has the property of long intervals if, there exists a sequence $(m(n)) \in \mathbb{N}^{\mathbb{N}}$ such that

$$\bigcup_{n\in\mathbb{N}} \{m(n), m(n)+1, \ldots, m(n)+n-1\} \in \mathcal{I}.$$

Note that Leonov introduced the property of long intervals for the dual filter, under the name the unbounded gap property. It can be shown that every dense P-ideal has PLI.

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Assume that $\mathcal{I}$ is an ideal with the property of long intervals. Then there exists a divergent series $\sum_n x_n$ in $\mathbb{R}$, with $x_n \not\to 0$, for which $\lambda(A(\mathcal{I})) = 1$. Consequently, $\lambda(A(\text{Fin})) = 0 < 1 = \lambda(A(\mathcal{I}))$.

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Final remark

Assume that a series $\sum x_n$ with terms in a Banach space is divergent (or is not unconditionally convergent).
Given a rasonable ideal $\mathcal{I}$ on $\mathbb{N}$ (e.g. analytic or coanalytic), we have not considered rearangements of the series from the measure viewpoint. Namely, we have not studied the measure size of the set $B := \left\{p \in P \colon \sum x_{p(n)} \text{ is } \mathcal{I}\text{-convergent}\right\}$.

The reason is that there is no Haar measure on the non-locally compact group $P = S_\infty$.
However, we can ask whether B is Haar null, that is whether there is a Borel probability measure μ on P such that $\mu(pBq) = 0$ for any $p, q \in P$. This is an open problem.

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