Permutation groups

Aleksandra Kwiatkowska

University of Wrocław / University of Münster
joint work with Maciej Malicki

September 19, 2018

Permutation groups - definition

X - countable set

Definition

$S_\infty = \mathrm{Sym}(X)$ – topological group of all bijections (=permutations) of X, equipped with the pointwise convergence topology

Permutation groups - definition

X - countable set

Definition

$S_\infty = \mathrm{Sym}(X)$ – topological group of all bijections (=permutations) of X, equipped with the pointwise convergence topology

Definition

A permutation group is a closed subgroup of S_∞.

Permutation groups - characterization

Let G be a separable completely metrizable topological group.

Proposition

The following conditions are equivalent:

1. *G is a permutation group;*
2. *G has a neighbourhood basis of the identity that consists of open subgroups;*
3. *G is an automorphism group of a countable first-order structure;*
4. *G is an automorphism group of a countable ultrahomogeneous first-order relational structure.*

Ultrahomogeneous structures

Definition

A countable first-order structure M is ultrahomogeneous if every automorphism between finite substructures of M can be extended to an automorphism of the whole M.

Ultrahomogeneous structures

Definition

A countable first-order structure M is ultrahomogeneous if every automorphism between finite substructures of M can be extended to an automorphism of the whole M.

Example

- rationals with the ordering
- the random graph
- the random poset
- the rational Urysohn space

Properties - extreme amenability

Definition

A topological group G is extremely amenable if every continuous action of G on a compact Hausdorff space has a fixed point.

Properties - extreme amenability

Definition

A topological group G is extremely amenable if every continuous action of G on a compact Hausdorff space has a fixed point.

- Examples: automorphism groups of rational numbers, ordered random graph, ordered rational Urysohn space
- Kechris-Pestov-Todorcevic: extreme amenability of $\mathrm{Aut}(M)$ (M-ultrahomogeneous) $\iff$ Ramsey Theorem holds for the family of finite substructures of M

Properties - ample generics

Definition

A topological group G has ample generics if for every n the diagonal conjugacy action of G on G^n given by $(g,(h_1,\ldots,h_n)) \mapsto (gh_1g^{-1},\ldots,gh_ng^{-1})$ has a comeager orbit.

Properties - ample generics

Definition

A topological group G has ample generics if for every n the diagonal conjugacy action of G on G^n given by $(g, (h_1, \ldots, h_n)) \mapsto (gh_1g^{-1}, \ldots, gh_ng^{-1})$ has a comeager orbit.

- Examples: automorphism groups of random graph, rational Urysohn space, countable atomless Boolean algebra
- Hrushovski property together with the free amalgamation property imply ample generics

Properties - topological similarity

Definition (Rosendal)

1. A tuple $(f_1, \ldots, f_n)$ in a topological group G is topologically similar to a tuple $(g_1, \ldots, g_n)$ if the map sending $f_i \mapsto g_i$ extends (necessarily uniquely) to a bi-continuous isomorphism between the topological groups generated by these tuples.

Properties - topological similarity

Definition (Rosendal)

1. A tuple $(f_1, \ldots, f_n)$ in a topological group G is topologically similar to a tuple $(g_1, \ldots, g_n)$ if the map sending $f_i \mapsto g_i$ extends (necessarily uniquely) to a bi-continuous isomorphism between the topological groups generated by these tuples.
2. The topological n-similarity class of $(f_1, \ldots, f_n)$ is then the set of the all the n-tuples in G that are topologically similar to it.

Properties - topological similarity

Definition (Rosendal)

1. A tuple $(f_1, \ldots, f_n)$ in a topological group G is topologically similar to a tuple $(g_1, \ldots, g_n)$ if the map sending $f_i \mapsto g_i$ extends (necessarily uniquely) to a bi-continuous isomorphism between the topological groups generated by these tuples.
2. The topological n-similarity class of $(f_1, \ldots, f_n)$ is then the set of the all the n-tuples in G that are topologically similar to it.

Remark

If two tuples are diagonally conjugate, then they are topologically similar.

Assumption

In this section, we will work with groups $\mathrm{Aut(M)}$ such that M has the property that algebraic closures of finite sets are finite.

That means: for every finite set $A \subseteq M$, the set

$$\{x \in M\colon \text{ the orbit of } x \text{ by } \mathrm{Aut}_A(M) \text{ is finite}\}$$

is finite, where
$\mathrm{Aut}_A(M) = \{f \in \mathrm{Aut}(M)\colon f(a) = a \text{ for every } a \in A\}$.

Results

Theorem (K.-Malicki)

Let M be a countable structure such that algebraic closures of finite sets are finite, let $G = Aut(M)$, and let $n \in \mathbb{N}$. Suppose that

- *there are comeagerly many n-tuples in G generating a non-precompact and non-discrete subgroup.*

Then each n-dimensional class of topological similarity in G is meager.

Corollary – the trichotomy

Corollary (K.-Malicki)

Let M be a countable structure such that algebraic closures of finite sets are finite, and let $G = Aut(M)$. Then for every n and n-tuple $\bar{f}$ in G:

1. $\langle \bar{f} \rangle$ *is precompact, or*
2. $\langle \bar{f} \rangle$ *is discrete, or*
3. *the similarity class of $\bar{f}$ is meager.*

The Hrushovski property

Definition

We say that a family of finite structures $\mathcal{K}$ has the Hrushovski property if for every $A \in \mathcal{K}$ and a tuple of partial automorphisms $(p_1, \ldots, p_n)$ of A there is $B \in \mathcal{K}$ with $A \subseteq B$ such that every p_i can be extended to an automorphism of B.

The Hrushovski property

Definition

We say that a family of finite structures $\mathcal{K}$ has the Hrushovski property if for every $A \in \mathcal{K}$ and a tuple of partial automorphisms $(p_1, \ldots, p_n)$ of A there is $B \in \mathcal{K}$ with $A \subseteq B$ such that every p_i can be extended to an automorphism of B.

Definition

An ultrahomogeneous structure M has the Hrushovski property if $\mathrm{Age}(M)$–the set of finite substructures of M, has the Hrushovski property.

Groups with ample generics

1. Case 1 of the trichotomy (precompact): groups Aut(M) such that M has the Hrushovsky property (for example M is the random graph, the triangle free random graph, the rational Urysohn space).
2. Case 2 of the trichotomy (discrete): homeomorphism group of the Cantor set.

Corollary - Hrushovski property

Let M be an ultrahomogeneous structure such that algebraic closures of finite sets are finite, and let $G = \mathrm{Aut}(M)$.

Corollary (K.-Malicki)

Suppose that G has ample generics, or just comeager n-similarity classes for every n. Then either

1. *M has the Hrushovski property, or*
2. *for every n the generic n-tuple $\bar{f}$ generates a discrete subgroup of G.*

Questions

Question

Is there a permutation group, which is extremely amenable and has ample generics?

Questions

Question

Is there a permutation group, which is extremely amenable and has ample generics?

Remark

There are Polish groups having simultaneously the two properties above. By a theorem of Pestov-Schneider, the group of measurable functions with values in S_∞, $L_0(S_\infty)$, is such.

Question

Is there a permutation group of a structure with a definable linear order, which has a comeager 2-conjugacy class?

Kechris-Rosendal criterion

Let M be an ultrahomogeneous structure and let $\mathcal{K} = \mathrm{Age}(M)$. Let

$$\mathcal{K}_n = \{(A, p_1^A, \ldots, p_n^A) \colon A \in \mathcal{K} \text{ and } p_i^A \text{ is a partial automorphism of } A\}$$

Theorem (Kechris-Rosendal)

There exists a comeager n-conjugacy class in $\mathrm{Aut}(M)$ iff $\mathcal{K}_n$ has n-JEP and n-WAP.

n-WAP

no 2-WAP:

Definition

There is $\bar{p} = (p_1, p_2)$ such that for every $\bar{q} = (q_1, q_2)$ and an embedding $\delta \colon \bar{p} \to \bar{q}$ there are embeddings $\alpha_1 \colon \bar{q} \to \bar{r}_1$ and $\alpha_2 \colon \bar{q} \to \bar{r}_2$ such that we cannot amalgamate $\bar{r}_1$ and $\bar{r}_2$ over $\bar{p}$. That is, there is no $\bar{s}$ and $\beta_1 \colon \bar{r}_1 \to \bar{s}$ and $\beta_2 \colon \bar{r}_2 \to \bar{s}$ such that $\beta_1 \circ \alpha_1 \circ \delta = \beta_2 \circ \alpha_2 \circ \delta$

What do we know

Theorem (K.-Malicki)

Automorphism groups of the following structures do not have 2-WAP:

1. *Ordered random graph. More generally, precompact Ramsey expansions of ultrahomogeneous directed graphs.*
2. *Ordered rational Urysohn space*
3. *Ordered random boron tree*

What do we know

Theorem (K.-Malicki)

Automorphism groups of the following structures do not have 2-WAP:

1. *Ordered random graph. More generally, precompact Ramsey expansions of ultrahomogeneous directed graphs.*
2. *Ordered rational Urysohn space*
3. *Ordered random boron tree*

Most of the structures above do not even have 1-WAP.

Ordered structures and comeager conjugacy classes

Truss proved that $(\mathbb{Q}, <)$ has a comeager conjugacy class. It seems that it was the only structure with a definable linear order, whose automorphism group has a comeager conjugacy class.

Ordered structures and comeager conjugacy classes

Truss proved that $(\mathbb{Q}, <)$ has a comeager conjugacy class. It seems that it was the only structure with a definable linear order, whose automorphism group has a comeager conjugacy class.

Theorem (K.-Malicki)

1. *The automorphism group of the ordered random poset has a comeager conjugacy class.*
2. *The automorphism group of the ordered random boron tree has a comeager conjugacy class.*

Definition

G-Polish group G

Definition

- The pair (g, h) cyclically generates G if $\{g^l h g^{-l} \colon l \in \mathbb{Z}\}$ is dense in G.
- In that case, we will say that G has a cyclically dense conjugacy class.

Definition

G-Polish group G

Definition

- The pair (g, h) cyclically generates G if $\{g^l h g^{-l} \colon l \in \mathbb{Z}\}$ is dense in G.
- In that case, we will say that G has a cyclically dense conjugacy class.

Remark

If G has a cyclically dense conjugacy class then G is topologically 2-generated.

Examples

Groups Aut(M) for the following countable structures M have a cyclically dense conjugacy class:

- Kechris-Rosendal: countable set (no structure), rational numbers, countable atomless Boolean algebra
- Macpherson: random graph
- Solecki: rational Urysohn space
- Glass-McCleary-Rubin: random poset
- Kaplan-Simon (2017): provided a model-theoretic condition for the existence of a cyclically dense conjugacy class.

Darji and Mitchell

Theorem (Darji-Mitchell)

Let G be the automorphism group of rationals or the automorphism group of a colored random graph.
Then for every non-identity $f \in G$, there is $g \in G$ such that (f, g) topologically 2-generates the whole group.

Strong$^+$ amalgamation property

Definition

Say that $\mathcal{F}$ satisfies the strong$^+$ amalgamation property if for every $A, B, C \in \mathcal{F}$, and embeddings $\phi_1 \colon A \to B$ and $\phi_2 \colon A \to C$ there is $D \in \mathcal{F}$ and there are embeddings $\psi_1 \colon B \to D$ and $\psi_2 \colon C \to D$ such that, denoting $S = \psi_1(B) \setminus \psi_1 \circ \phi_1(A)$ and $T = \psi_2(C) \setminus \psi_2 \circ \phi_2(A)$ we have:

1. $\psi_1 \circ \phi_1 = \psi_2 \circ \phi_2$,
2. for every $x \in S$ and $y \in T$, we have $x \neq y$,
3. for every n and every relation symbol $R \in L$ of arity n, for all n-tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ of elements in $S \cup T$ such that
   - $x_i \in S$ iff $y_i \in S$ for $i = 1, \ldots, n$,
   - $\{x_1, \ldots, x_n\} \cap S \neq \emptyset$ and $\{x_1, \ldots, x_n\} \cap T \neq \emptyset$,

   we have $R^D(x_1, \ldots, x_n)$ iff $R^D(y_1, \ldots, y_n)$.

Remarks

Remark

If L consists only of relations of arity 2, (3) above says that: For every relation symbol $R \in L$ and for all tuples (x_1, x_2) and (y_1, y_2) such that either we have $x_1, y_1 \in S$ and $x_2, y_2 \in T$ or we have $x_1, y_1 \in T$ and $x_2, y_2 \in S$, it holds that: $R^D(x_1, x_2)$ iff $R^D(y_1, y_2)$.

Remarks

Remark

If L consists only of relations of arity 2, (3) above says that: For every relation symbol $R \in L$ and for all tuples (x_1, x_2) and (y_1, y_2) such that either we have $x_1, y_1 \in S$ and $x_2, y_2 \in T$ or we have $x_1, y_1 \in T$ and $x_2, y_2 \in S$, it holds that: $R^D(x_1, x_2)$ iff $R^D(y_1, y_2)$.

Remark

If $\mathcal{F}$ has the free amalgamation property then it has the strong$^+$ amalgamation property.

More examples

Example

Strong$^+$ amalgamation property holds for

1. every countable structure that has the free amalgamation property (e.g., the random graph or the random triangle free graph),
2. the universal tournament

Results

Theorem (K.-Malicki)

Let M be a relational ultrahomogeneous structure that $\mathrm{Age}(M)$ has the strong$^+$ amalgamation property. Then for every $n = 1, 2, \ldots$, $Aut(M)^n$ as well as $L_0(Aut(M))$ have a cyclically dense conjugacy class. In fact, each of these groups is cyclically generated by a pair generating the free group.

Results

Theorem (K.-Malicki)

Let M be a relational ultrahomogeneous structure that $\mathrm{Age}(M)$ has the strong$^+$ amalgamation property. Then for every $n = 1, 2, \ldots$, $Aut(M)^n$ as well as $L_0(Aut(M))$ have a cyclically dense conjugacy class. In fact, each of these groups is cyclically generated by a pair generating the free group.

The same conclusion holds for the rational Urysohn space, the random poset, the rational numbers, the countable atomless Boolean algebra, the countable atomless Boolean algebra equipped with the dyadic measure.