The Open Graph Dichotomy and the second level of the Borel hierarchy.

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- An example is the **complete** D-hypergraph on X: the complement of $\Delta^D(X)$.

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- A κ-**coloring** of H is a map $c : X \to \kappa$ such that $c^{-1}(\{i\})$ is H-independent for all $i \in \kappa$, for κ a cardinal.
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- There is a κ-coloring of H iff there is a homomorphism from H to the complete hypergraph on a set of cardinality κ.
- The **chromatic number** of H, $\chi(H)$, is the least κ such that H has a κ-coloring.
- If H is the complete D-hypergraph on X then $\chi(H) = |X|$.

The box-open hypergraph dichotomy

For $t \in D^{<\mathbb{N}}$, note $N_t := \{x \in D^{\mathbb{N}} \mid t \sqsubseteq x\}$. Define

$$\mathbb{H}_{D^{\mathbb{N}}} := \bigcup_{t \in D^{<\mathbb{N}}} \prod_{d \in D} N_{t^\frown(d)}.$$

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Let Γ be a class of spaces.

$\mathrm{OGD}^D(\Gamma)$

If H is a box-open D-hypergraph on $X \in \Gamma$, exactly one holds

- $\chi(H) \leq \aleph_0$
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We denote the latter case by $\mathbb{H}_{D^{\mathbb{N}}} \leq_c H$.

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Theorem (Feng)

For $|D| \geq 2$, $\mathrm{OGD}^D(\boldsymbol{\Sigma}^1_1)$ *holds.*

OGD and AD

It turns out that the box-open hypergraph dichotomy follows from determinacy.

Theorem

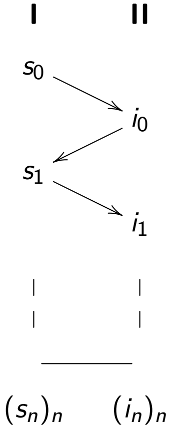
Assume AD*. If H is a box-open $\mathbb{N}$-hypergraph on Y analytic Hausdorff, $X \subseteq Y$, exactly one holds*

- $\chi(H \upharpoonright X) \leq \aleph_0$.
- $\mathbb{H}_{\mathbb{N}^{\mathbb{N}}} \leq_c H \upharpoonright X$.

Let us sketch a proof of this theorem.
Without loss of generality, we suppose that $Y = \mathbb{N}^{\mathbb{N}}$.

A game for OGD

Here H is a $\mathbb{N}$-hypergraph on $\mathbb{N}^{\mathbb{N}}$, $X \subseteq \mathbb{N}^{\mathbb{N}}$, $s_n \in \mathbb{N}^{<\mathbb{N}}$ and $i_n = 0, 1$



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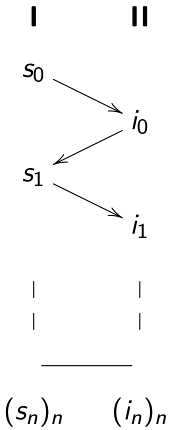

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If $i_m = 1$ then **I** must play $s_n \sqsupset s_m$ for all $n > m$.

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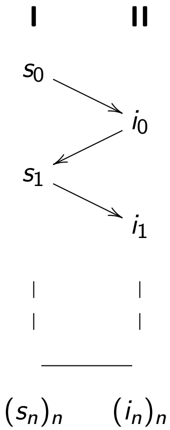

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Then $(s_n)_n$ is $\sqsubseteq$-cofinal in $x \in \mathbb{N}^{\mathbb{N}}$, and

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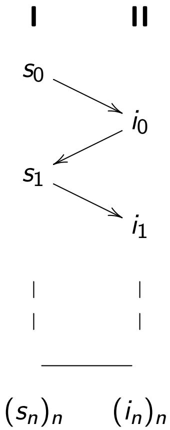

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Case 2. $i_n = 0$ for all rounds $n > m$.

I wins if $\prod_{n \geq m} N_{s_n} \subseteq H$.

I has a winning strategy implies $\mathbb{H}_{\mathbb{N}^\mathbb{N}} \leq_c H \upharpoonright X$.

- H is a $\mathbb{N}$-hypergraph on $\mathbb{N}^\mathbb{N}$, $X \subseteq \mathbb{N}^\mathbb{N}$.
- **I** builds $(s_n)_n \subseteq \mathbb{N}^{<\mathbb{N}}$ and **II** builds $(i_n)_n \in 2^\mathbb{N}$.
- The rule: if $i_m = 1$ then **I** must play $s_n \sqsupset s_m$ for all $n > m$.

Case 1. $\exists^\infty n\,(i_n = 1)$

I wins if $\bigcup_n s_n = x \in X$.

Case 2. $\forall n > m\,(i_n = 0)$

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- Suppose that **I** has a wining strategy φ.
- Identify each $y \in \mathbb{N}^\mathbb{N}$ with a sequence $(i_n)_n$ as in Case 1.
- For any such y, **I** plays $\varphi(y) \in X$.

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- By construction φ is continuous.
- Using Case 2, φ is a homomorphism from $\mathbb{H}_{\mathbb{N}^{\mathbb{N}}}$ to $H \restriction X$

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- $x \mapsto t_x$ is constant on a neighborhood $N_{c(x)}$ of x.
- Using Case 2, c is an $\aleph_0$-coloring of H. □

First application: a dichotomy for K_σ spaces.

Theorem (Hurewicz, Kechris - Saint Raymond)

Suppose $\mathrm{OGD}^{\mathbb{N}}(\Gamma)$. *Given* $Y \subseteq X$, $Y \in \Gamma$, *exactly one holds:*

- *Y is contained in a K_σ subset of X*
- *there is a closed cont. injection* $\varphi : \mathbb{N}^{\mathbb{N}} \to X$ *ranging in* Y.

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- H_Y is box-open in $Y^{\mathbb{N}}$.
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- H_Y is box-open in $Y^{\mathbb{N}}$.
- If $\chi(H_Y) \leq \aleph_0$ then Y is contained in a K_σ set.
- Otherwise, $\mathrm{OGD}(\Gamma)$ gives us φ ...

It is continuous and injective, to see that it is closed, take $(x_n)_n$ in $\mathbb{N}^{\mathbb{N}}$ and suppose $\varphi(x_n) \to y$.
If $(x_n)_n$ has no convergent subsequence, it contains a subsequence of a $\mathbb{H}_{\mathbb{N}^{\mathbb{N}}}$-hyperedge, but then $\left(\varphi(x_n)\right)_n$ cannot converge. □

Second application: the Hurewicz dichotomy

- Denote $\mathbb{N}^{\mathbb{N}}_*$ the space $\mathbb{N}^{\mathbb{N}} \cup \{s \frown (\infty) \mid s \in \mathbb{N}^{<\mathbb{N}}\}$.
- Equipped with the **smallest** topology making both $\{t\}$ and $\mathcal{N}_t = \{s \in \mathbb{N}^{\mathbb{N}}_* \mid t \sqsubseteq s\}$ clopen.

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- It is homeomorphic to the Cantor space.
- The Baire space $\mathbb{N}^{\mathbb{N}}$ is a G_δ subset that is not F_σ.
- A map $f : X \to Y$ **reduces** $A \subseteq X$ to $B \subseteq Y$ iff $f^{-1}(B) = A$.

Theorem (Hurewicz, Kechris-Louveau-Woodin)

Assume $\mathsf{OGD}^{\mathbb{N}}(\Gamma)$. *Given $A \subseteq X$, $A \in \Gamma$, exactly one holds:*

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Take H_A the following hypergraph on A:

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So $\chi(H_A) \leq \aleph_0$ implies A is F_σ.

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For $y \neq x$, we have $y_s = x_s$, otherwise by looking at $\left(\varphi(s \frown (n) \frown x_n)\right)_n$ for $x_{2n} = x$ and $x_{2n+1} = y$ we would have a contradiction.

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Then

$$\bar{\varphi} : \mathbb{N}^{\mathbb{N}}_* \longrightarrow X$$
$$s \longrightarrow \begin{cases} \varphi(s) & \text{if } s \in \mathbb{N}^{\mathbb{N}} \\ x_s & \text{otherwise.} \end{cases}$$

is a continuous reduction from $\mathbb{N}^{\mathbb{N}}$ to A. □

A third application: the Jayne-Rogers theorem

OGD also gives a generalisation of the Jayne-Rogers theorem.

Recall that a function is σ-**continuous with closed witnesses** if it can be covered by countably many continuous functions with closed domains.

Theorem

Assume $\mathrm{OGD}^{\mathbb{N}}(\Gamma)$*. For* $X \in \Gamma$ *and* $f : X \to Y$ *Borel, the following are equivalent:*

- f *is* σ*-continuous with closed witnesses,*
- f *is* G_δ*-measurable.*

The original Jayne-Rogers theorem is the case $\Gamma = \mathbf{\Sigma}^1_1$.

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Theorem

Assume $\mathrm{OGD}^{\mathbb{N}}(\Gamma)$*. For* $X \in \Gamma$ *and* H *a box-open hereditary* $\mathbb{N}$*-hypergraph on* X*, either* $\chi(H) \leq \aleph_0$ *of* $\chi(H) \geq \mathfrak{d}$*.*

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Hereditary hypergraphs

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Assume $\mathrm{OGD}^{\mathbb{N}}(\Gamma)$. *For* $X \in \Gamma$ *and* H *a box-open hereditary* $\mathbb{N}$*-hypergraph on* X*, either* $\chi(H) \leq \aleph_0$ *of* $\chi(H) \geq \mathfrak{d}$.

Sketch of proof. $\mathbb{H}_{\mathbb{N}^{\mathbb{N}}}$ is not hereditary, so let's look at a hereditary version. Call $(\mathbb{N})^{\mathbb{N}}$ the injective sequences of $\mathbb{N}^{\mathbb{N}}$.

$$\mathbb{H}'_{\mathbb{N}^{\mathbb{N}}} = \{(s \frown (i_n) \frown \mathbf{b}(n))_{n \in \mathbb{N}} \mid s \in \mathbb{N}^{<\mathbb{N}}, (i_n)_n \in (\mathbb{N})^{\mathbb{N}}, \mathbf{b} \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}\}$$

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Since $A \subseteq \mathbb{N}^\mathbb{N}$ is $\mathbb{H}'_{\mathbb{N}^\mathbb{N}}$-independent iff $\overline{A}$ is compact, $\chi(\mathbb{H}'_{\mathbb{N}^\mathbb{N}}) = \mathfrak{d}$. Notice now that if H is hereditary and $\mathbb{H}_{\mathbb{N}^\mathbb{N}} \leq_c H$, then $\mathbb{H}'_{\mathbb{N}^\mathbb{N}} \leq_c H$. □

The general case

- The **covering number** of a σ-ideal I on a set X, denoted by $\mathrm{cov}(I)$, is the least cardinality of a family of elements of I covering X.
- Call $\mathcal{M}$ the ideal of meager sets.

Theorem

Assume $\mathrm{OGD}^D(\Gamma)$. *For* $X \in \Gamma$ *and* H *a box-open* D*-hypergraph on* X*, either* $\chi(H) \leq \aleph_0$ *of* $\chi(H) \geq \mathrm{cov}(\mathcal{M})$.

Since $\chi(\mathbb{H}_{\mathbb{N}^\mathbb{N}}) = \mathrm{cov}(\mathcal{M})$, this bound cannot be strengthened in the general case. However..

Stronger bounds in the case D finite.

Notice that $\mathbb{H}_{2^{\mathbb{N}}}$ is the complete graph on $2^{\mathbb{N}}$, so the case $D = 2$ is trivial.

Theorem

Assume $\mathrm{OGD}(\Gamma)$. *For $X \in \Gamma$ and G an open graph on X, either $\chi(H) \leq \aleph_0$ of $\chi(H) \geq \mathfrak{c}$.*

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Theorem

Assume OGD(Γ). *For* $X \in \Gamma$ *and* G *an open graph on* X, *either* $\chi(H) \leq \aleph_0$ *of* $\chi(H) \geq \mathfrak{c}$.

Call $\mathcal{N}$ the ideal of null sets.
Call $\mathfrak{b}$ the least cardinality of a family $\mathcal{F} \subseteq \mathbb{N}^{\mathbb{N}}$ such that for all $c \in \mathbb{N}^{\mathbb{N}}$ there is a $d \in \mathcal{F}$ that is not eventually dominated by c.

Theorem

For $2 \leq D < \aleph_0$, *assume* OGD(Γ). *For* $X \in \Gamma$ *and* H *an box-open* D*-hypergraph on* X, *either* $\chi(H) \leq \aleph_0$ *of* $\chi(H) \geq \mathrm{cov}(\mathcal{N}) \cdot \mathfrak{b}$.

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Thank you!