Large Cardinals in the Stable Core

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joint work with Sy Friedman and Victoria Gitman

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- Canonical inner models
- The Stable Core
- Large cardinals in the Stable Core

Classical canonical inner models

What is a canonical inner model?

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Example

Gödel's constructible universe L:

- $L_0 = \emptyset$,
- $L_{\alpha+1} =$definable subsets of L_α,
- $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ for limit ordinals λ, and
- $L = \bigcup_{\alpha\in\mathrm{Ord}} L_\alpha$.

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$L[\mu]$, the canonical inner model for a measurable cardinal.

Again, $L[\mu]$ misses a lot of large cardinals and can be very far from V, e.g. if V has two measurable cardinals.

HOD is not canonical

Definition

- OD = the collection of all sets definable with ordinal parameters.
- HOD = the collection of all hereditarily ordinal definable sets, i.e.

$$\mathrm{HOD} = \{a \mid \mathrm{trcl}(\{a\}) \subseteq \mathrm{OD}\}.$$

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How close is HOD to V?

Is V a class forcing extension of HOD?

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Theorem (Vopenka)

Every set of ordinals is set-generic over HOD*: If* A *is a set of ordinals, then there is a partial order* $\mathbb{P} \in \mathrm{HOD}$ *and a* HOD*-generic filter* $G \subseteq \mathbb{P}$ *such that* $\mathrm{HOD}[A] = \mathrm{HOD}[G]$.

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There is a definable class S *such that every initial segment of* S *is in* HOD *and* V *is a class forcing extension of* (HOD, S)*.*

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The Stability Predicate S

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Definition

The *Stability Predicate* S consists of all triples (α, β, n) such that

- α and β are n-good cardinals, and
- $H_\alpha \prec_{\Sigma_n} H_\beta$.

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Theorem (Friedman)

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Idea: Use Jensen coding.

GCH can fail everywhere in the Stable Core

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Theorem (Friedman, Gitman, M.)

Suppose $\mathbb{P} \in L$ is a forcing notion and $G \subseteq \mathbb{P}$ is L-generic. Then there is a further forcing extension $L[G][H]$ such that $G \in L[S^{L[G][H]}]$.

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The following can consistently happen in the Stable Core:

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With class forcing we can even get:

Theorem (Friedman, Gitman, M.)

It is consistent that GCH fails at all regular cardinals in the Stable Core.

Digression: Measurable cardinals and $L[\mu]$

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How does the Stable Core of $L[\mu]$ look like? Can it have a measurable cardinal?

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Theorem (Kennedy, Magidor, Väänänen)

- *If* $0^\sharp$ *exists, then* $0^\sharp \in L[\mathrm{Card}]$.
- *If there is a measurable cardinal, then* $L[\mu] \subseteq L[\mathrm{Card}]$.
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The structure of $L[\mathrm{Card}]$ becomes regular in the presence of large cardinals.

Theorem (Kennedy, Magidor, Väänänen, Welch)

Assume there is (a little more than) a measurable limit of measurables, then in $L[\mathrm{Card}]$*: There are no measurable cardinals and* GCH *holds.*

Measurable cardinals in the Stable Core

Some of their techniques generalize to the Stable Core.

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This generalizes to sequences $(\mu^{(\xi)} \mid \xi < \nu)$ for normal measures $\mu^{(\xi)}$ on distinct measurable cardinals $\kappa^{(\xi)}$ with $\nu < \kappa^{(0)}$. But:

Question

Can the Stable Core have a measurable limit of measurable cardinals?

Coding into the Stable Core over $L[\mu]$

Theorem (Friedman, Gitman, M.)

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Theorem (Friedman, Gitman, M.)

It is consistent that the Stable Core has a measurable cardinal and the GCH fails at all regular cardinals.

Measurable cardinals are not downward absolute to the Stable Core

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Theorem (Kunen)

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Weakly compact cardinals are not downward absolute.

Theorem (Friedman, Gitman, M.)

It is consistent that a cardinal κ is measurable in V, but not even weakly compact in the Stable Core.

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Question

What does the Stable Core look like in the presence of large cardinals?

- Is there a bound on the large cardinals the Stable Core can have?
- Or: Are large cardinals downward absolute to the Stable Core?
- Does the GCH hold?

"There is an ever changing list of questions in set theory the answers to which would greatly increase our understanding of the universe of sets. The difficulty of course is the ubiquity of independence: almost always the questions are independent."

(W. H. Woodin in Suitable Extender Models I)

Thank you for your attention!