The role of ideals in topological selection principles

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joint work with Jaroslav Šupina

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Basic terms

Definition

The family I $P(!)$ is called **_L09e**, if it has properties:

(I1) $8\ 2\ I,\ ?\ \ 8\ \ !\ \ ?\ 2\ I,$

(I2) $8;?\ 2\ I\ \ !\ \ 8\ [\ ?\ 2\ I,$

(I3) $!\ 2\!\!/\ I,$

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- $I, \ J$ denote ideals on $!$.

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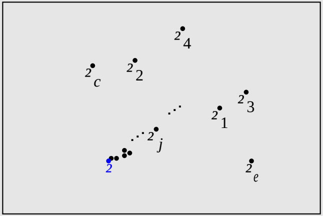
A sequence $h²_j\ :\ j\ 2\ !\ i$ elements of a topological space ± is **I]Hnj®0}X0j§** to $²\ 2$ ± if the set $fj\ 2\ !\ :\ ²_j\ 2\ ª g\ 2\ I$ for each neighborhood $ª$ of $²$, (written $²_j \overset{I}{!}\ ²$).



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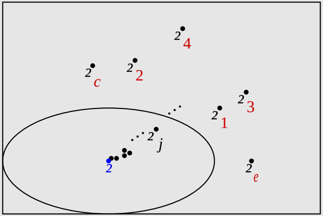

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- Let $hT_j : j\ 2\ !\,i$ be a sequence of functions on $\pm$ and T being function on $\pm$.
- $T_j \overset{I}{!}\ T$, $fj\ 2\ ! : jT_j(^2)\ \ T(^2)j\ \ "g\ 2\ I$ for each $^2\ 2\ \pm$ and for each $" > 0$.

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- Let $\cdot$ denote constant zero-value function on $\pm$.

$$. = \overset{\mathrm{n}}{} 8 \; 2 \; ^! (\mathrm{G}_q(\pm) \, n \, f \cdot g) : \; \cdot \; 2 \; \overline{f \acute{} : (9j \; 2 \; !) \; 8(j) = \acute{} g} \overset{\mathrm{o}}{} :$$

$$I\text{-} \; . = f 8 \; 2 \; ^! (\mathrm{G}_q(\pm) \, n \, f \cdot g) : \; 8 \text{ is } I\text{-convergent to } \cdot g :$$

- We use $.$ instead of S_j- $..$

Selection principles

Let P and R be families of sets.

- ± has $\begin{smallmatrix} P \\ R \end{smallmatrix}$ or ± is a $[P; R]$-space if for every $hq_j : j\ 2\ !\, i\ 2\ P$ there is $hj_h : h\ 2\ !\, i$ such that $hq_{j_h} : h\ 2\ !\, i\ 2\ R$.

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  - If P and R denote convergences then ± is a $[P_q; R_q]$-space if for every $hq_j : j\ 2\ !\, i$ such that $q_j \overset{P}{!}\ q$ there is $hj_h : h\ 2\ !\, i$ such that $q_{j_h} \overset{R}{!}\ q$.

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- ± has $\frac{P}{R}$ or ± is a $[P; R]$-space if for every $hq_j : j\ 2\ !i\ 2\ P$ there is $hj_h : h\ 2\ !i$ such that $hq_{j_h} : h\ 2\ !i\ 2\ R$.
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- ± is an $\sim_1(P; R)$-space if for a sequence $hU_j : j\ 2\ !i$ of elements of P we can select a set ${}^{a}{}_{j}\ 2\ U_j$ for each $j\ 2\ !$ such that $h{}^{a}{}_{j} : j\ 2\ !i$ is a member of R.

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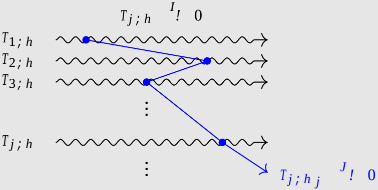
$\sim_1(I\text{-}\ .; J\text{-}\ .)$ can be imagined in the following way



$\sim_1(I\text{-} \quad .; J\text{-} \quad .)$

classical convergence) I-convergence

$\sim_1(I\text{-}\ \ .;\ J\text{-}\ \ .)$

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Observation

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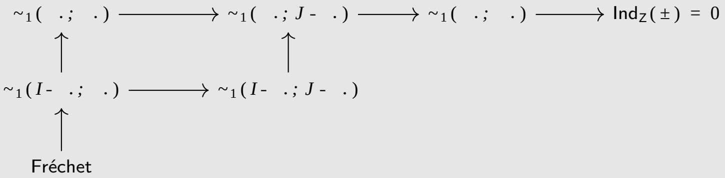

Diagram. Selection principles for functions.

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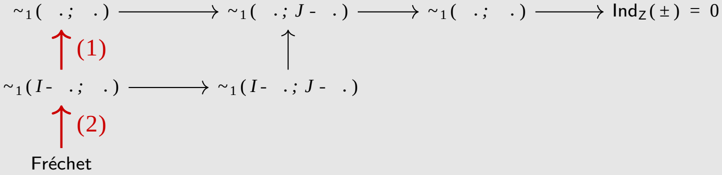

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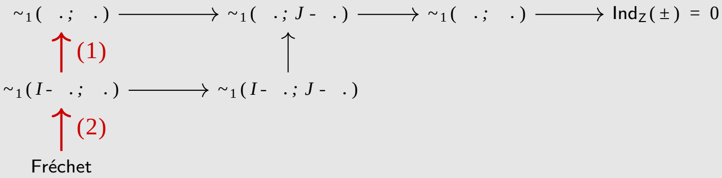

Diagram. Selection principles for functions.

- $\sim_1(\ .;\ .)$-space , [.; .]-space , Fréchet space

(1) The relation between $\sim_1(I- \quad .; \quad .)$-space and $\sim_1(\quad .; \quad .)$.

Proposition (V. Š., J. Šupina)

d0§ ± @0 9 §nqnenX_H9e q9H0u ¦ \0j $\mathrm{G}_q(\pm)$ _ *9j* $\sim_1(I] \quad .; \quad .)]$ *q9H0 _T 9jL nje´ _T* $\mathrm{G}_q(\pm)$
\9 $^{I]}_{\ .}\cdot$ *9jL* $\sim_1(\quad .; \quad .)$*u*

- In general, $\sim_1(I- \quad .; J- \quad .) \ ) \ \frac{I-}{J-} \cdot$ for arbitrary ideals I, J.

$\sim_1(I$- .; J- .)

(2) The relation between I- . and . .

Lemma

Sn} 9j ´ Hn«j§9@e0 T9 h _e´ nT T«jH§_nj E nj ± «H\ §\9§ · 2 $\overline{E\ n\ f \cdot g}$ 9jL _§ @_a0H§_®0 0j« h 0}9§_nj hT$_j$: j 2 ! i §\0}0 _ 9j _L09e I «H\ §\9§ T$_j$ ^{I}! ·u

Theorem (V. Š., J. Šupina)

d0§ ± @0 9 ¦´H\njn¸ §nqnenX_H9e q9H0u ¦\0 Tneen°_jX §9§0 h 0j§ 9}0 0x«_®9e0j§u

r9s ± _ 9j $\sim_1$(.; .)] q9H0u

r@s $G_q(\pm)$ _ 9j $\sim_1(I]$.; .)] q9H0 Tn} 0®0}´ _L09e Iu

rHs $G_q(\pm)$ \9 $^{I]}$ · Tn} 0®0}´ _L09e Iu

P R and $\sim_{\mathrm{o}}(P; R)$

P, R as covers.

- denotes the family of all open $!$-covers of $\pm$.
- denotes the family of all open -covers of $\pm$.

$\frac{P}{R}$ and ~$_o(P; R)$

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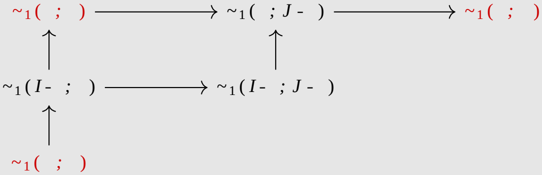

Diagram. Covering selection principles.

$\frac{P}{R}$ and $\sim_0(P; R)$

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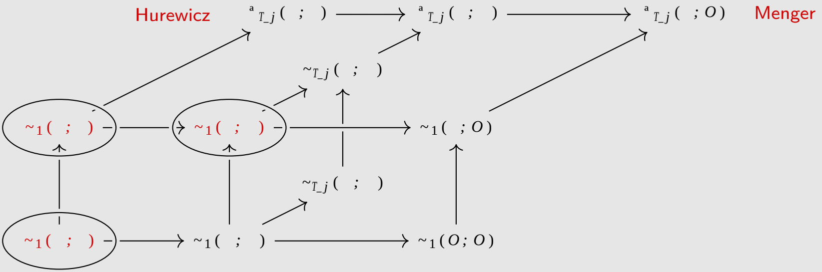

Diagram. Scheepers' diagram (1996).

$\frac{P}{R}$ and $\sim_0(P; R)$

- We say that a sequence $h\Gamma_j : j\ 2\ !\, i$ is **hnjn§nj0 0x«0jH0** if for any $j\ 2\ !$ and $^2\ 2\ \pm$ we have $\Gamma_j(^2) \quad \Gamma_{j+1}(^2)$.
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- I- $\overset{h}{.} = f 8 \; 2 \; ^{!} (\mathrm{G}_q(\pm) \, n \, f \cdot g) : \; 8$ is I-monotone and I-convergent to $\cdot g:$

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- $I\text{-}\ \overset{h}{.} = f8\ 2\ ^!(\mathrm{G}_q(\pm)\, n\ f \cdot g) : 8$ is I-monotone and I-convergent to $\cdot g:$

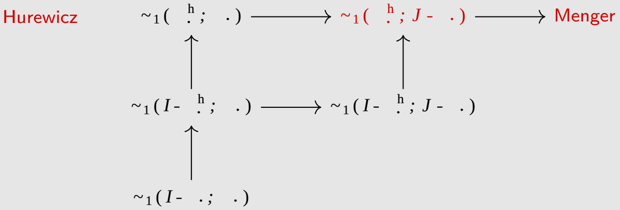

Diagram. Monotonic selection principles for functions.

$\frac{P}{R}$ and $\sim_o(P; R)$

- M. Scheepers (1997):

  Hurewicz $\sim_1($ $\overset{h}{.}$; $.)$

- P. Szewczak, B. Tsaban (2017):

  Hurewicz) *J* -Hurewicz) Menger.

Proposition (V. Š, J. Šupina)

^T ± _ 9 q0}TOH§e´ jn}h9e §nqnenX_H9e q9HO §\0j §\0 Tneen°_jX 9}0 0x«_®9e0j§u g n}0n®0}J
_T ± _ 9}@_§}9}´ §nqnenX_H9e q9HO §\0j r9s r@su

r9s $G_q(\pm)$ \9 $\overset{h}{.}$ J w i . u

r@s $G_q(\pm)$ \9 §\0 q}nq0}§´ $\sim_1($ $\overset{h}{.}$; J] .)u

rHs ± qn 0 0 9 J] [«}0° _H¶ q}nq0}§´u

- L. Bukovský, P. Das and J. Šupina (2017): the ideal version of Scheepers' result.

$$\sim_1(I\text{-}\ \ ; J\text{-}\ \ )\ \ !\ \ \sim_1(I\text{-}\ \ ^{\backslash}; J\text{-}\ \ )\ \ ,\ \ \sim_1(I\text{-}\ \ .; J\text{-}\ \ .)\ \ !\ \ \sim_1(I\text{-}\ \ \overset{\text{h}}{.}; J\text{-}\ \ .):$$

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$$\sim_1(I\text{-}\ \ ; J\text{-}\ \ ) \ ! \ \sim_1(I\text{-}\ \ \backslash; J\text{-}\ \ ) \ , \ \sim_1(I\text{-}\ \ .; J\text{-}\ \ .) \ ! \ \sim_1(I\text{-}\ \ \overset{h}{.}; J\text{-}\ \ .):$$

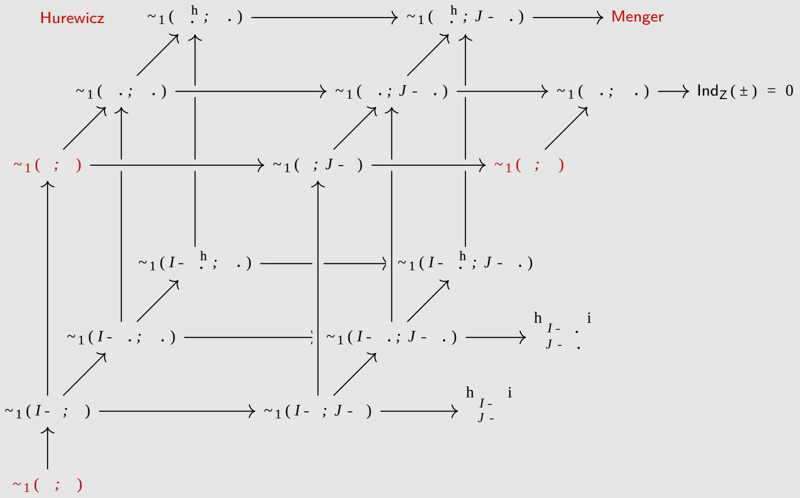

Diagram. The overall relations of investigated properties.

Cardinal invariants

- $jnj(\sim_1(I\text{-}\;\; ; J\text{-}\;\;)$-space) denotes the minimal cardinality of a perfectly normal space which is not an $\sim_1(I\text{-}\;\; ; J\text{-}\;\;)$-space.

Cardinal invariants

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- M. Hrušák, F. Hernández

  Hn® (I) = h_j $fjAj$: A I ^ (8~ 2 [!]$^!$)(98 2 A) j~ \ 8j = !g:

  - p = h_j f : (9 an ideal I) Hn® (I) = g:
- M. Repický (2018)

  $k_{I;J}$ = h_j $fjAj$: A I ^ A 6 $_b$ J g:

  - if I is tall then $k_{I;S\_j}$ = Hn® (I).

Let A $P(\omega)$.

- a sequence $\in {}^{\omega}A$ will be called an A- **e9enh**.

Let $A \quad P(!)$.

- a sequence $\quad 2 \; {}^{!}A$ will be called an A- **e9enh**.
- a function $' \; 2 \; {}^{!}! \;\; J$-**Xn0** through A-slalom $\quad$ if $fj : \; '(j) \; 2 \;\; (j)g \; 2 \; J^{L}$, i.e., $fj : \; '(j) \; 2 \; ! \; n \;\; (j)g \; 2 \; J$.
    - We say that $'$ goes through I-slalom instead of $'$ S_j-goes through I-slalom.

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$$\mathrm{b} = \mathrm{h\_j}\ fjRj:\ R\ ^{!}\ !;\ (8\mathrm{S\_j}\text{-slalom }\ )(9\ '\ 2\ R)\ :(\ '\ \text{goes through }\ )g:$$

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$$(I;J) = \mathrm{h\_j}\ \overset{\mathrm{n}}{}\ jRj:\ R\ \text{contains}\ I^{L}\text{-slaloms},\ (8\ '\ 2\ {}^{!}!)(9\ \ 2\ R)\ :(\ '\ J\text{-goes through}\ \ )\overset{\mathrm{o}}{}\ :$$

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$$(I;J)\ =\ \mathrm{h\_j}\ \overset{\mathrm{n}}{}\ jRj:\ R\text{ contains }I^{L}\text{-slaloms, }(8\ '\ 2\ ^{!}!)(9\ \ 2\ R)\ :(\ '\ J\text{-goes through }\ )\overset{\mathrm{o}}{}\ :$$

- J. Šupina's results (2016):
  - $(\mathrm{S\_j};J)\ =\ \mathrm{b}_{J}$,
  - if $I_1\ \ _b\ I_2$ and $J_1\ \ _{b\ ?}\ J_2$ then $(I_2;J_1)\ \ \ (I_1;J_2)$,
  - $\mathrm{jnj}(\sim_1(I\text{- };J\text{- })\ =\ \ (I;J))$.

Theorem (V. Š., J. Šupina)

ros ^T I 6 b J §\0j (I; J) h_j fk I ; J ; b J gu

r©s ^T I 6 b J 9jL J b I §\0j (I; J) = h_j fk I ; J ; (J ; J)gu

r¨s ^T I _ §9ee §\0j (I; S_j) = h_j fHn® (I); bgu

Theorem (V. Š., J. Šupina)

ros ^T I 6 b J §\0j (I ; J) h_j fk$_{I;J}$;b$_J$ gu

r©s ^T I 6 b J 9jL J b I §\0j (I ; J) = h_j fk$_{I;J}$; (J ; J)gu

r¨s ^T I _ §9ee §\0j (I ; S_j) = h_j fHn® (I); bgu

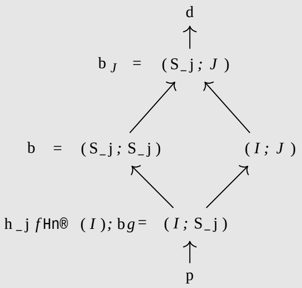

Diagram. Cardinal (I ; J).

Critical cardinality

- Let K being a discrete topological space.

$$jKj < \ (I;J) \ , \ G_q(K) \text{ has } \genfrac{}{}{0pt}{}{I\text{-} \ \ \cdot}{sJ \ w \ i \ .} \ , \ G_q(K) \text{ has } \sim_1(I\text{-} \ \ .;J\text{-} \ \ .)$$

$$, \ G_q(K) \text{ has } \sim_1(I\text{-} \ \ \overset{h}{.};J\text{-} \ \ .):$$

- A. Kwela–M. Repický (2018)

$$jKj < \mathtt{Hn®} \ (I) \ , \ G_q(K) \text{ has } \genfrac{}{}{0pt}{}{I \ w \ i \ .}{w \ i \ .} \ , \ G_q(K) \text{ has } \genfrac{}{}{0pt}{}{I\text{-} \ \ \cdot}{.} \ , \ K \text{ has the property } {}^{I\text{-}} \ \ .$$

- Let K being a discrete topological space.

$$jKj < \quad (I; J) \ , \ G_q(K) \text{ has } \ I\text{-} \ . \ sJ \ w \ i \ . \ , \ G_q(K) \text{ has } \sim_1(I\text{-} \ .; J\text{-} \ .)$$

$$, \ G_q(K) \text{ has } \sim_1(I\text{-} \ h; J\text{-} \ .):$$

- A. Kwela–M. Repický (2018)

$$jKj < \text{Hn®}\ (I) \ , \ G_q(K) \text{ has } \ I \ w \ i \ . \ w \ i \ . \ , \ G_q(K) \text{ has } \ I\text{-} \ . \ , \ K \text{ has the property } \ I\text{-} \ .$$

Corollary (V. Š., J. Šupina)

- $d0§ \ I; J \quad P(!) \ @0 \ \_L09e \ u$

ros $jnj(\sim_1(I] \ .; J] \ .)) = jnj(\sim_1(I] \ h; J] \ .)) = jnj(\ I] \ J \ w \ i \ . \ ) = \ (I; J)u$

r©s $jnj(\sim_1(\ .; J] \ .)) = jnj(\sim_1(\ h; J] \ .)) = jnj(\ J \ w \ i \ . \ ) = b_J \ u$

- $\hat{}T \ I \ \_ \ §9ee \ §\backslash 0j$

r¨s $jnj(\sim_1(I] \ ; \ )) = jnj(\sim_1(I] \ .; \ .)) = jnj(\sim_1(I] \ h; \ .)) = jnjr \ I] \ w \ i \ . \ s =$ $h\_j f\text{Hn®}\ (I); bgu$

rVs **r8u b °0e9Ś g u |0q_Hc‹s** $jnjr \ I \ w \ i \ . \ w \ i \ . \ s = jnjr \ I] \ . \ s = jnjr \ I] \ s = \text{Hn®}\ (I)u$

Critical cardinality

Proposition

ros Let ± ! !. If $G_q(\pm)$ has the property $\sim_1(I\text{-}\ \overset{h}{.}; J\text{-}\ .)$, then ± is bounded in $(^! !; \ ^J)$.

r©s Let I be a tall ideal. If A I has $^{I-}$ or $G_q(A)$ has $^{I-}$. or $^{I\,w\,i}_{w\,i}$. then A has a pseudounion.

r¨s Let I be a tall ideal. If A $I \setminus [!]^!$ and $G_q(A)$ is an $\sim_1(I\text{-}\ .;\ .)$ then A has a pseudounion and the family of increasing enumerations of its elements is bounded in $(^! !;\ )$.

if	there is a set ± of reals of cardinality such that:
b $_J$ c	$G_q(\pm)$ does not have the property $\sim_1(I\text{-}\ \overset{h}{.}; J\text{-}\ .)$ $G_q(\pm)$ does not have the property $\sim_1(I\text{-}\ .; J\text{-}\ .)$
Hn® (I) c	$G_q(\pm)$ does not have the property $^{I-}$. $G_q(\pm)$ does not have the property $^{I\,w\,i}_{w\,i}$. ± does not have $^{I-}$
h_j fHn® (I); bg c	$G_q(\pm)$ does not have the property $\sim_1(I\text{-}\ .;\ .)$ ± does not have $\sim_1(I\text{-}\ ;\ )$.

Critical cardinality

- Consistency

*	b = c	)	jnj(~$_1$(I- ;)) = Hn® (I) for every tall ideal I
*	b < Hn® (I)	)	jnj(~$_1$(I- ;)) < Hn® (I) for every tall ideal I
	p = b	)	jnj(~$_1$(I- ;)) = b
*	Hn® (I) < b	)	jnj(~$_1$(I- ;)) < b
	b$_J$ < d	)	jnj(~$_1$(I- ; J-)) < d

Table: jnj(~$_1$(I- ;)) regarding to cardinal consistency.

* We can reformulate for ~$_1$(I- .; J- .)-space and its monotone version.

Critical cardinality

- Consistency

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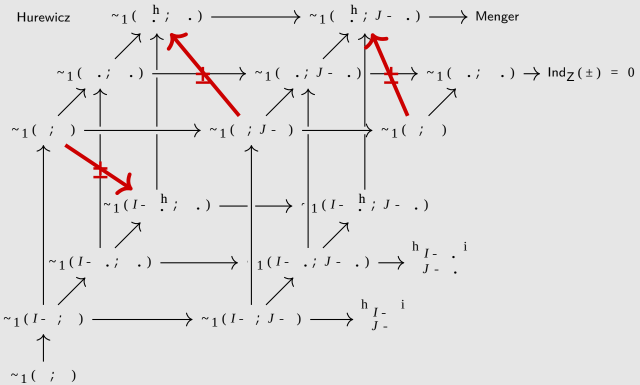
Table: jnj(~$_1$(I- ;)) regarding to cardinal consistency.

* We can reformulate for ~$_1$(I- .; J- .)-space and its monotone version.

condition	± is	G$_q$(±) is not
p < b	~$_1$(;)-space	~$_1$(U- $\dot{}^h$; .)-space
Hn® (I) < b	~$_1$(;)-space	~$_1$(I- $\dot{}^h$; .)-space
b < b$_U$	~$_1$(; U-)-space	~$_1$($\dot{}^h$; .)-space
b$_J$ < d	~$_1$(;)-space	~$_1$($\dot{}^h$; J- .)-space
b < Hn® (I)	[I- ;]-space	~$_1$(I- $\dot{}^h$; .)-space

Critical cardinality

condition	± is	$G_q(\pm)$ is not
p < b	~1(;)-space	~1(U- h. ; .)-space
Hn® (I) < b	~1(;)-space	~1(I- h. ; .)-space
b < b_U	~1(; U-)-space	~1(h. ; .)-space
b_J < d	~1(;)-space	~1(h. ; J- .)-space
b < Hn® (I)	[I- ;]-space	~1(I- h. ; .)-space



Thank you for your attention

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