Fluctuations of spectrally negative Markov additive processes

Andreas E. Kyprianou[1] and Zbigniew Palmowski[2,3]

[1] Department of Mathematical Sciences, The University of Bath
Claverton Down, Bath, UK
[2] Mathematical Institute, University of Wrocław
pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
[3] Mathematical Institute, Utrecht University
P.O. Box 80.010, 3508 TA Utrecht, The Netherlands
e-mails: a.kyprianou@bath.ac.uk, zpalma@math.uni.wroc.pl

Summary. For spectrally negative Markov Additive Processes (MAPs) we generalize classical fluctuation identities developed in Zolotarev (1964), Takács (1967), Bingham (1975), Suprun (1976), Emery (1973), Rogers (1990) and Bertoin (1997) which concern one and two sided exit problems for spectrally negative Lévy processes.

1 Spectrally Negative Markov Additive Processes

This paper presents some fluctuation identities for a special, but none the less quite general, class of Markov Additive Processes (MAP). Before entering our discussion on the subject we shall simply begin by defining the class of processes we intend to work with and its properties.

Following Asmussen and Kella (2000) we consider a process $X(t)$, where $X(t) = X^{(1)}(t) + X^{(2)}(t)$, and the independent processes $X^{(1)}(t)$ and $X^{(2)}(t)$ are specified by the characteristics: $q_{ij}, G_{ij}, \sigma_i, a_i, \nu_i(dx)$ which we shall now define. Let $J(t)$ be a right-continuous, ergodic, finite state space continuous time Markov chain, with states $\mathcal{I} = \{1, \ldots, N\}$, and with intensity matrix $\mathbf{Q} = (q_{ij})$. We denote the jumps of the process $J(t)$ by $\{T_i\}$ (with $T_0 = 0$). Let $\{U_n^{(ij)}\}$ be i.i.d. random variables, which are also independent of J, with distribution function $G_{ij}(\cdot)$ ($U^{(ii)} \equiv 0$). Define the jump process by

$$X^{(1)}(t) = \sum_{n\geq 1} \sum_{i,j} U_n^{(ij)} \mathbf{1}_{\{J(T_{n-1})=i,\ J(T_n)=j,\ T_n \leq t\}}.$$

For each $i \in \mathcal{I}$, let $X^i(t)$ be a Lévy process, independent of all other stochastic quantities, with Laplace exponent

$$\log E(\exp \alpha X^i(1)) = \psi_i(\alpha) = a_i\alpha + \frac{\sigma_i^2\alpha^2}{2} + \int_{-\infty}^{0} \big(e^{\alpha y} - 1 - \alpha y 1_{(-1,0)}(y)\big)\nu_i(dy),$$

where $1_{(-1,0)}(y)$ is valued 1 if $y \in (-1,0)$ and valued 0 otherwise and $\int_{-\infty}^{0}(1 \wedge |y|^2)\nu_i(dy) < \infty$. By $X^{(2)}(t)$ we denote the process which behaves in law like $X^i(t)$, when $J(t) = i$. Note that each of the measures ν_i are supported on $(-\infty, 0)$ as well as the distributions of each $U^{(ij)}$ and in this respect we say that X is a *spectrally negative* MAP.

Letting $\mathbf{Q} \circ \widehat{\mathbf{G}}(\alpha) = (q_{ij}\widehat{G}_{ij}(\alpha))$, where $\widehat{G}_{ij}(\alpha) = E\left(\exp(\alpha U^{(ij)})\right)$, we define *matrix cumulant generating function* of MAP $X(t)$:

$$\mathbf{F}(\alpha) = \mathbf{Q} \circ \widehat{\mathbf{G}}(\alpha) + \mathrm{diag}(\psi_1(\alpha), \ldots, \psi_N(\alpha)) \ . \tag{1}$$

Note then that $\mathbf{F}(\alpha)$ is well defined and finite at least for $\alpha \geq 0$. Within this regime of α, Perron-Frobenius theory identifies $\mathbf{F}(\alpha)$ as having a real-valued eigenvalue with maximal absolute value which we shall label $\kappa(\alpha)$. The corresponding left and right $1 \times N$ eigenvectors we label $\mathbf{v}(\alpha)$ and $\mathbf{h}(\alpha)$ respectively. In this text we shall always write vectors in their horizontal form and use the usual $^{\mathrm{T}}$ to mean transpose. Since $\mathbf{v}(\alpha)$ and $\mathbf{h}(\alpha)$ are given up to multiplying constants, we are free to normalize them such that

$$\mathbf{v}(\alpha)\,\mathbf{h}(\alpha)^{\mathrm{T}} = 1 \text{ and } \pi\mathbf{h}(\alpha)^{\mathrm{T}} = 1\ ,$$

where $\pi = \mathbf{v}(0)$ is the stationary distribution of J. Note also that $\mathbf{h}(0) = \mathbf{e}$, the $1 \times N$ vector consisting of a row of ones. We shall write $h_i(\alpha)$ for the i-th element of $\mathbf{h}(\alpha)$. The eigenvalue $\kappa(\alpha)$ is a convex function (this can also be easily verified) such that $\kappa(0) = 0$, $\kappa(\infty) = \infty$ and $\kappa'(0)$ is the asymptotic drift of X in the sense that for each $i \in \mathcal{I}$ we have $\lim_{t\uparrow\infty} E(X(t)|J(0) = i, X(0) = x)/t = \kappa'(0)$. The sign of $\kappa'(0)$ also determines the asymptotic behaviour of X. When $\kappa'(0) > 0$, the process drifts to infinity, $\lim_{t\uparrow\infty} X(t) = \infty$, when $\kappa'(0) < 0$, the process drifts to minus infinity, $\lim_{t\uparrow\infty} X(t) = -\infty$, and when $\kappa'(0) = 0$ the process oscillates, $\limsup_{t\uparrow\infty} X(t) = -\liminf_{t\uparrow\infty} X(t) = \infty$. For the right inverse of κ we shall write Φ on $[0,\infty)$. That is to say, for each $q \geq 0$,

$$\Phi(q) = \sup\{\alpha \geq 0 : \kappa(\alpha) = q\}.$$

Note that the properties of κ imply that $\Phi(q) > 0$ for $q > 0$. Further $\Phi(0) = 0$ if and only if $\kappa'(0) \geq 0$ and otherwise $\Phi(0) > 0$.

We shall assume the afore mentioned class of MAPs are defined on a probability space with probabilities $\{\mathbb{P}_{i,x} : i \in \mathcal{I}, x \in \mathbb{R}\}$ and right-continuous natural filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$. It can be checked that under the following Girsanov change of measure

$$\left.\frac{d\mathbb{P}^{\gamma}_{i,x}}{d\mathbb{P}_{i,x}}\right|_{\mathcal{F}_t} := e^{\gamma(X(t)-x)-\kappa(\gamma)t}\frac{h_{J(t)}(\gamma)}{h_i(\gamma)}, \text{ for } \gamma \text{ such that } \kappa(\gamma) < \infty, \tag{2}$$

the process $(X, \mathbb{P}^{\gamma}_{i,x})$ is again a spectrally negative MAP whose intensity matrix $\mathbf{F}_\gamma(\alpha)$ is well defined and finite for $\alpha \geq -\gamma$; see for example Palmowski and Rolski (2002). If $\mathbf{F}_\gamma(\alpha)$ has largest eigenvalue $\kappa_\gamma(\alpha)$ and associated right eigenvector $\mathbf{h}_\gamma(\alpha)$, the triple $(\mathbf{F}_\gamma(\alpha), \kappa_\gamma(\alpha), \mathbf{h}_\gamma(\alpha))$ is related to the original triple $(\mathbf{F}(\alpha), \kappa(\alpha), \mathbf{h}(\alpha))$ via

$$\mathbf{F}_\gamma(\alpha) = \mathbf{\Delta_h}(\gamma)^{-1}\mathbf{F}(\alpha+\gamma)\mathbf{\Delta_h}(\gamma) - \kappa(\gamma)\mathbf{I} \text{ and } \kappa_\gamma(\alpha) = \kappa(\alpha+\gamma) - \kappa(\gamma) \ , \tag{3}$$

where $\mathbf{I}$ is the $N \times N$ identity matrix and

$$\mathbf{\Delta_h}(\gamma) := \operatorname{diag}(h_1(\gamma), ..., h_N(\gamma)) .$$

We shall also use a similar definition for the matrix $\mathbf{\Delta_v}(\gamma)$.

As much as possible we shall prefer to work with matrix notation. For a random variable Y and (random) time τ, we shall understand $\mathbf{E}_x(Y; J(\tau))$ to be the matrix with (i,j)-th elements $\mathbb{E}_{i,x}(Y; J(\tau) = j)$. For an event, A, $\mathbf{P}_x(A; J(\tau))$ will be understood in a similar sense. For simplicity we shall follow the tradition that $\mathbf{E}(\cdot) = \mathbf{E}_0(\cdot)$ and $\mathbf{P}(\cdot) = \mathbf{P}_0(\cdot)$. For shorthand we will denote $\mathbf{I}_{ij}(q) = \mathbb{P}_{i,0}(J(\mathbf{e}_q) = j)$, in other words $\mathbf{I}(q) = q(q\mathbf{I} - \mathbf{Q})^{-1}$.

These details and more concerning the basic characterization of MAPs can be found in Chapter XI of Asmussen (2003).

2 Time reversal

Predominant in the forthcoming discussion will be the use of the bivariate process $(\widehat{J}, \widehat{X})$, representing the process (J, X) time reversed from a fixed moment in the future when $J(0)$ has the stationary distribution π. For definitiveness, we mean

$$\widehat{J}(s) = J((t-s)^-) \text{ and } \widehat{X}(s) = X(t) - X\left((t-s)^-\right), \ 0 \leq s \leq t$$

under $\mathbb{P}_{\pi,0} = \sum_{i \in \mathcal{I}} \pi_i \mathbb{P}_{i,0}$. The characteristics of $(\widehat{J}, \widehat{X})$ will be indicated by using a hat over the existing notation for the characteristics of (J, X). For example $\widehat{\mathbf{F}}$, $\widehat{\mathbf{h}}$, $\widehat{\kappa}$ and so on. To relate these characteristics to the original ones, recall that the intensity matrix of $\widehat{J}$ must satisfy

$$\widehat{\mathbf{Q}} = \mathbf{\Delta}_\pi^{-1}\mathbf{Q}^{\mathrm{T}}\mathbf{\Delta}_\pi \ ,$$

where $\mathbf{\Delta}_\pi$ is the diagonal matrix whose entries are given by the vector π. Hence according to (1) we find that when it exists

$$\widehat{\mathbf{F}}(\alpha) = \mathbf{\Delta}_\pi^{-1}\mathbf{F}(\alpha)^{\mathrm{T}}\mathbf{\Delta}_\pi.$$

Since $(\mathbf{v}(\alpha)\mathbf{F}(\alpha))^{\mathrm{T}} = \mathbf{F}(\alpha)^{\mathrm{T}}\mathbf{v}(\alpha)^{\mathrm{T}} = \kappa(\alpha)\mathbf{v}(\alpha)^{\mathrm{T}}$ we have that

$$\widehat{\mathbf{F}}(\alpha)\mathbf{\Delta}_\pi^{-1}\mathbf{v}(\alpha)^{\mathrm{T}} = \mathbf{\Delta}_\pi^{-1}\mathbf{F}(\alpha)^{\mathrm{T}}\mathbf{\Delta}_\pi\mathbf{\Delta}_\pi^{-1}\mathbf{v}(\alpha)^{\mathrm{T}} = \kappa(\alpha)\mathbf{\Delta}_\pi^{-1}\mathbf{v}(\alpha)^{\mathrm{T}}$$

showing that $\widehat{\kappa}(\alpha) \geq \kappa(\alpha)$. On the other hand a similar calculation reveals that

$$\widehat{\kappa}(\alpha)\, \mathbf{\Delta}_\pi \widehat{\mathbf{h}}(\alpha)^{\mathrm{T}} = \mathbf{\Delta}_\pi \widehat{\mathbf{F}}(\alpha)\, \widehat{\mathbf{h}}(\alpha) = \mathbf{F}(\alpha)^{\mathrm{T}}\, \mathbf{\Delta}_\pi \widehat{\mathbf{h}}(\alpha)^{\mathrm{T}}$$

so that $\widehat{\kappa}(\alpha) \leq \kappa(\alpha)$ and hence $\widehat{\kappa} = \kappa$ and $\mathbf{\Delta}_\pi \widehat{\mathbf{h}}(\alpha)^{\mathrm{T}} = \mathbf{v}(\alpha)^{\mathrm{T}}$.

Instead of talking about the process $(\widehat{J}, \widehat{X})$ we shall talk about the process (J, X) under probabilities $\{\widehat{\mathbb{P}}_{i,0} : i \in \mathcal{I}\}$ meaning the MAP whose characteristics are given by $\widehat{\mathbf{F}}$. Note also for future use, following classical time reversed path analysis, for $y \geq 0$,

$$\begin{aligned}\mathbb{P}_{i,0}\left(-I(t) \in dy | J(t) = j\right) &= \mathbb{P}_{j,0}\left(\widehat{S}(t) - \widehat{X}(t) \in dy | \widehat{J}(t) = i\right) \\ &= \widehat{\mathbb{P}}_{j,0}\left(S(t) - X(t) \in dy | J(t) = i\right), \qquad (4)\end{aligned}$$

where $I(t) = \inf_{0\leq s\leq t} X(s)$, $S(t) = \sup_{0\leq s\leq t} X(s)$ and $\widehat{S}(t) = \sup_{0\leq s\leq t} \widehat{X}(s)$. (A diagram may help to explain the last identity). Asmussen (1989, 2000) gives a more thorough discussion on time reversal.

3 The intensity matrix $\mathbf{\Lambda}(q)$

Also important for the main results of this paper will be a brief summary of the classical analysis of first passage upward with the help of exponential change of measure given in (2).

Define for each $x \geq 0$

$$\tau_x^+ := \inf\{t \geq 0 : X(t) \geq x\}.$$

Note that for each $q \geq 0$ and $x \geq 0$,

$$\mathbf{\Delta}_{\mathbf{h}}(\Phi(q))^{-1}\mathbf{E}\left(e^{\Phi(q)x - q\tau_x^+} 1_{(\tau_x^+ < \infty)}; J(\tau_x^+)\right)\mathbf{\Delta}_{\mathbf{h}}(\Phi(q)) = \mathbf{I}^{\Phi(q)}(\tau_x^+).$$

Here

$$\mathbf{I}_{ij}^{\Phi(q)}(\tau_x^+) = \mathbb{P}_{i,0}^{\Phi(q)}(J(\tau_x^+) = j, \tau_x^+ < \infty) = \mathbb{P}_{i,0}^{\Phi(q)}(J(\tau_x^+) = j),$$

where the last equality follows since $(X, \mathbb{P}^{\Phi(q)})$ drifts to infinity on account of the fact that $\kappa'_{\Phi(q)}(0) = \kappa'(\Phi(q))$ which is strictly positive by convexity of κ. Hence

$$\mathbf{E}\left(e^{-q\tau_x^+} 1_{(\tau_x^+ < \infty)}; J(\tau_x^+)\right) = e^{-\Phi(q)x}\mathbf{\Delta}_{\mathbf{h}}(\Phi(q))\mathbf{I}^{\Phi(q)}(\tau_x^+)\mathbf{\Delta}_{\mathbf{h}}(\Phi(q))^{-1}.$$

A little thought reveals that process $\{J(\tau_x^+) : x \geq 0\}$ is again an ergodic Markov chain whenever X does not drift to $-\infty$. Let us suppose that under $\mathbf{P}^{\Phi(q)}$, for $q \geq 0$, its intensity matrix is given by $\mathbf{\Lambda}(q)$ so that $\mathbf{I}^{\Phi(q)}(\tau_x^+) = \exp(\mathbf{\Lambda}(q)x)$. We thus obtain the following result.

Theorem 1. *For $q \geq 0$,*

$$\mathbf{E}\left(e^{-q\tau_x^+}1_{(\tau_x^+<\infty)}; J(\tau_x^+)\right) = \boldsymbol{\Delta}_{\mathbf{h}}(\Phi(q))e^{-(\Phi(q)\mathbf{I}-\boldsymbol{\Lambda}(q))x}\boldsymbol{\Delta}_{\mathbf{h}}(\Phi(q))^{-1}. \tag{5}$$

Unfortunately, it does not seem that an explicit expression for $\boldsymbol{\Lambda}(q)$ can be derived from existing literature. Indeed establishing an expression for $\boldsymbol{\Lambda}(q)$ for spectrally negative MAPs is an open problem. Recent progress has been made however by Miyazawa and Takada (2002) (see also the references therein) who consider a special case of the spectrally negative MAP that we have here. That is, the case when the jump part of each of the processes X^i are of bounded variation and finite mean. For this class, they prove that $\boldsymbol{\Lambda}(0)$ solves $\mathbf{F}(-\boldsymbol{\Lambda}(0)) = \mathbf{0}$ where we should understand the latter to mean that

$$-\boldsymbol{\Delta}_{\mathbf{a}}\boldsymbol{\Lambda}(0) + \boldsymbol{\Delta}_{\sigma^{\mathbf{2}}/\mathbf{2}}\boldsymbol{\Lambda}(0)^2 + \int_{-\infty}^{0}\boldsymbol{\Delta}_{\nu}(dy)(e^{-\boldsymbol{\Lambda}(0)y} - \mathbf{I} + \boldsymbol{\Lambda}(0)y1_{[-1,0]}(y))$$
$$+\int_{-\infty}^{0}\mathbf{Q}\circ\mathbf{G}(dy)e^{-\boldsymbol{\Lambda}(0)y} = \mathbf{0}\ , \tag{6}$$

where $\boldsymbol{\Delta}_{\mathbf{a}}$ is the diagonal matrix with entries $a_1, ..., a_N$ along the diagonal, similarly $\boldsymbol{\Delta}_{\sigma^{\mathbf{2}}/\mathbf{2}}$ is diagonal with elements $\sigma_1^2/2, ..., \sigma_N^2/2$, matrix $\boldsymbol{\Delta}_{\nu}(\cdot)$ is diagonal with $\nu_1(\cdot), \ldots, \nu_N(\cdot)$ on the diagonal and the matrix $\mathbf{G}$ has entries $G_{ij}(\cdot)$ corresponding to the distributions of each U_{ij}. It can be seen from the straightforward martingale techniques described by Pistorius (2006) however, that in general, one may characterize $\boldsymbol{\Lambda}(q)$ as a solution to the matrix equation $\mathbf{F}_{\Phi(q)}(-\boldsymbol{\Lambda}(q)) = \mathbf{0}$. (We are grateful to M. Pistorius for pointing this out). For further discussion concerning uniqueness of the solution see Rogers (1994), Asmussen (2000) and Miyazawa and Takada (2002).

4 Main results

In this article we shall establish fluctuation identities concerning the exit times τ_a^+ for $a \geq 0$ (defined at the beginning of the previous section) and

$$\tau_0^- := \inf\{t \geq 0 : X(t) \leq 0\}.$$

Our results will be expressed in terms of $N \times N$ matrix functions $\mathbf{W}^{(q)}(x)$ and $\mathbf{M}^{(q)}(x)$ both of which are mappings from $\mathbb{R}$ to $[0, \infty)$ with a parameter range $q \geq 0$.

Before moving to the main results, we shall devote a little time to establishing some further notation. Here and throughout we work with the definition that $\mathbf{e}_q$ is random variable which is exponentially distributed with mean $1/q$ and independent of (J, X).

For $q > 0$, let

$$\widehat{\mathbf{D}}(q) = \mathbf{\Delta_v}\left(\Phi\left(q\right)\right)(\Phi(q)\mathbf{I} - \widehat{\mathbf{\Lambda}}(q))^{-1}\mathbf{\Delta_v}\left(\Phi\left(q\right)\right)^{-1},$$

where the matrix $\widehat{\mathbf{\Lambda}}(q)$ takes the same definition as $\mathbf{\Lambda}(q)$ but for the time reversed process $\widehat{X}$. We shall also be interested in the following related limiting quantity

$$\mathbf{D} := \lim_{q\downarrow 0} q\widehat{\mathbf{D}}(q)^{\mathrm{T}}\mathbf{I}(q)^{-1}.$$

There are two cases to consider.

When X drifts to $-\infty$ we know that $\Phi(0) > 0$ then simply

$$\mathbf{D} = \mathbf{\Delta_v}\left(\Phi\left(0\right)\right)^{-1}(\Phi(0)\mathbf{I} - \widehat{\mathbf{\Lambda}}(0)^{\mathrm{T}})^{-1}\mathbf{\Delta_v}\left(\Phi\left(0\right)\right)\mathbf{Q}.$$

On the other hand, when X does not drift to ∞, denote by π^+ the stationary distribution of $\{J(\tau_x^+), x \geq 0\}$ under $\widehat{\mathbf{P}}$. Let $\mathbf{\Pi}^+ = \mathbf{e}^{\mathrm{T}}\,\pi^+$. By Coolen-Schrijner and van Doorn (2002) there exists a matrix

$$\mathbf{A} = \int_0^\infty \left(e^{\widehat{\mathbf{\Lambda}}(0)t} - \mathbf{\Pi}^+\right)\,dt$$

called a deviation matrix (or Drazin inverse) of $\widehat{\mathbf{\Lambda}}(0)$. It solves uniquely the equations

$$-\widehat{\mathbf{\Lambda}}(0)\mathbf{X} = \mathbf{X}\widehat{\mathbf{\Lambda}}(0), \quad \mathbf{X}(-\widehat{\mathbf{\Lambda}}(0))\mathbf{X} = \mathbf{X} \quad \text{and} \quad (-\widehat{\mathbf{\Lambda}}(0))\mathbf{X}(-\widehat{\mathbf{\Lambda}}(0)) = -\widehat{\mathbf{\Lambda}}(0).$$

Note that $\Phi(0) = 0$ and hence $\mathbf{\Delta_v}(\Phi(0)) = \Delta_\pi$. Also

$$\begin{aligned} q\widehat{\mathbf{D}}(q)^{\mathrm{T}}\mathbf{I}(q)^{-1} &\sim \mathbf{\Delta}_\pi^{-1}[(\Phi(q)\mathbf{I} - \widehat{\mathbf{\Lambda}}(q))^{-1}]^{\mathrm{T}}\mathbf{\Delta}_\pi(q\mathbf{I} - \mathbf{Q})\mathbf{\Delta}_\pi^{-1}\mathbf{\Delta}_\pi \\ &= \mathbf{\Delta}_\pi^{-1}\left[(q\mathbf{I} - \widehat{\mathbf{Q}})(\Phi(q)\mathbf{I} - \widehat{\mathbf{\Lambda}}(q))^{-1}\right]^{\mathrm{T}}\mathbf{\Delta}_\pi\,, \end{aligned}$$

where $\mathbf{B}(q) \sim \mathbf{C}(q)$ means that $\mathbf{B}(q)\mathbf{C}(q)^{-1} \to \mathbf{I}$ as $q \to 0$. Morover,

$$\begin{aligned} (q\mathbf{I} - \widehat{\mathbf{Q}})(\Phi(q)\mathbf{I} - \widehat{\mathbf{\Lambda}}(q))^{-1} &= (q\mathbf{I} - \widehat{\mathbf{Q}})\int_0^\infty e^{\widehat{\mathbf{\Lambda}}(q)t}e^{-\Phi(q)t}\,dt \\ &\sim q\mathbf{A} - \widehat{\mathbf{Q}}\mathbf{A} + q\int_0^\infty e^{-\Phi(q)t}\mathbf{\Pi}^+\,dt \\ &\quad - \int_0^\infty e^{-\Phi(q)t}\widehat{\mathbf{Q}}\mathbf{e}^{\mathrm{T}}\pi^+\,dt \\ &= q\mathbf{A} - \widehat{\mathbf{Q}}\mathbf{A} + \frac{q}{\Phi(q)}\mathbf{\Pi}^+ \\ &\sim \kappa'(0)\mathbf{\Pi}^+ - \widehat{\mathbf{Q}}\mathbf{A}. \end{aligned}$$

Hence

$$\mathbf{D} = \mathbf{\Delta}_\pi^{-1}\left[\kappa'(0)\mathbf{\Pi}^+ - \widehat{\mathbf{Q}}\,\mathbf{A}\right]^{\mathrm{T}}\mathbf{\Delta}_\pi.$$

We now give the main results which are uniformly given under the following assumption.

Assumption 2 *None of the processes X^i are downward subordinators.*

The assumption that none of the processes X^i are downward subordinators offers the convenience that matrices such as $\widehat{\mathbf{\Lambda}}(q)$ have no zero columns.

Theorem 3. *For each $q \geq 0$ there exist $N \times N$ matrix functions $\mathbf{W}^{(q)}(\cdot)$ and $\mathbf{M}^{(q)}(\cdot)$ such that the following hold (for convenience we shall write $\mathbf{W}$ for $\mathbf{W}^{(\mathbf{0})}$).*

(i) *The matrix $\mathbf{W}^{(q)}$ almost everywhere differentiable on $(0, \infty)$, equal to zero on $(-\infty, 0]$ and satisfies*

$$\mathbf{W}^{(q)}(x) = \mathbf{\Delta_h}(\Phi(q)) e^{\Phi(q)x} \mathbf{W}_{\Phi(q)}(x) \mathbf{\Delta_h}(\Phi(q))^{-1}$$

for all $x \in \mathbb{R}$ where $\mathbf{W}_{\Phi(q)}$ plays the role of $\mathbf{W}$ under $\mathbb{P}^{\Phi(q)}$.

(ii) *The matrix $\mathbf{M}^{(q)}$ is characterized by its Laplace transform on $(0, \infty)$,*

$$\int_{[0,\infty)} e^{-\beta x} \mathbf{M}^{(q)}(dx) = (\mathbf{F}(\beta) - q\mathbf{I})^{-1}(\mathbf{I} - \beta \widehat{\mathbf{D}}(q)^T)(q\mathbf{I} - \mathbf{Q}) \tag{7}$$

for sufficiently large β and it is equal to $\mathbf{I}$ on $(-\infty, 0]$.

(iii) *For $x \geq 0$,*

$$\mathbf{E}_x \left(e^{-q\tau_0^-} 1_{(\tau_0^- < \infty)}; J(\tau_0^-) \right) = \mathbf{M}^{(q)}(x). \tag{8}$$

(iv) *For $x \leq a$,*

$$\mathbf{E}_x \left(e^{-q\tau_a^+} 1_{(\tau_a^+ < \tau_0^-)}; J(\tau_a^+) \right) = \mathbf{W}^{(q)}(x)\, \mathbf{W}^{(q)}(a)^{-1}.$$

(v) *For $x \leq a$,*

$$\mathbf{E}_x \left(e^{-q\tau_0^-} 1_{(\tau_a^+ > \tau_0^-)}; J(\tau_0^-) \right) = \mathbf{M}^{(q)}(x) - \mathbf{W}^{(q)}(x) \mathbf{W}^{(q)}(a)^{-1} \mathbf{M}^{(q)}(a).$$

To some extent the results in Theorem 3 generalize known expressions for spectrally negative Lévy processes established over a number of years by Zolotarev (1964), Takács (1967), Emery (1973) Bingham (1975), Suprun (1976), Rogers (1990) and Bertoin (1997).

Aside from being a natural generalization of the known results for spectrally negative Lévy processes, there are genuine reasons for wanting to know the conclusions presented in Theorems 3. In recent years classical models in the theory of financial mathematics, risk and queues have been replaced by ones involving Markov modulation and general classes of Lévy processes. In the latter case one particular class that has proved to be quite successful in this respect is spectrally negative Lévy processes, on account of the robustness of their fluctuation theory; see for example Schmidli (1999), Asmussen (2000), Asmussen and Kella (2000), Asmussen et al. (2004), Avram et al. (2004), Klüppelberg et al. (2004), Pistorius (2003), Huzak et al. (2004), Chan (2005), Chiu and Yin (2005) and Dube et al. (2004) to name but a few. Our results justify the reasoning that, in general, one should hope to be able to work with spectrally negative MAPs in these models (see e.g. Miyazawa (2004)).

5 Asmussen-Kella martingale

The basis of this section is the new martingale introduced by Asmussen and Kella (2000). This martingale is closely related to other martingales of an exponential type found in Kella and Whitt (1992) and Jacod and Shiryaev (1987) for example. For our purposes we shall simply introduce the Asmussen-Kella martingale in the form that we shall use it; however the reader is urged to return to the original presentation in Asmussen and Kella (2000) in order to appreciate that it can take a more general form. The martingale in question is zero mean, vector valued and takes the form

$$\int_0^t e^{-\beta Z(u)}\mathbf{1}_{J(u)}du \cdot \mathbf{F}(\beta) + e^{-\beta Z(0)}\mathbf{1}_{J(0)} - e^{-\beta Z(t)}\mathbf{1}_{J(t)} - \beta \int_0^t \mathbf{1}_{J(u)}dS(u)\;,$$

where $\mathbf{1}_i$ is the $1 \times N$ vector whose elements are zero except at the i-th position where it is 1, $\beta \geq 0$ and $Z(t) = S(t) - X(t)$. From this martingale we shall deduce an identity, below, which forms the basis of our proofs.

Theorem 4. *For* $\beta \geq 0$

$$\begin{aligned}&\mathbf{E}\left(e^{\beta I(\mathbf{e}_q)}; J(\mathbf{e}_q)\right)^T (\mathbf{F}(\beta) - q\mathbf{I})^T \\ &\quad = q\mathbf{\Delta_v}(\Phi(q))[\beta(\Phi(q)\mathbf{I} - \widehat{\mathbf{\Lambda}}(q))^{-1} - \mathbf{I}]\mathbf{\Delta_v}(\Phi(q))^{-1}\;, \qquad (9)\end{aligned}$$

where the $N \times N$ *matrix* $\widehat{\mathbf{\Lambda}}(q)$ *is the intensity matrix of the process* $\{J(\tau_x^+) : x \geq 0\}$ *under* $\widehat{\mathbf{P}}^{\Phi(q)}$.

Proof From (4) we have that

$$\begin{aligned}\mathbb{E}_{i,0}\left(e^{\beta I(\mathbf{e}_q)}1_{(J(\mathbf{e}_q)=j)}\right) &= \mathbb{E}_{i,0}\left(e^{\beta I(\mathbf{e}_q)}|J(\mathbf{e}_q) = j\right)\mathbb{P}_{i,0}\left(J(\mathbf{e}_q) = j\right) \\ &= \widehat{\mathbb{E}}_{j,0}\left(e^{-\beta Z(\mathbf{e}_q)}|J(\mathbf{e}_q) = i\right)\widehat{\mathbb{P}}_{j,0}\left(J(\mathbf{e}_q) = i\right)\frac{\pi_j}{\pi_i} \\ &= \pi_i^{-1}\widehat{\mathbb{E}}_{j,0}\left(e^{-\beta Z(\mathbf{e}_q)}1_{(J(\mathbf{e}_q)=i)}\right)\pi_j.\end{aligned}$$

That is to say, the matrix $\widehat{\mathbf{E}}\left(e^{-\beta Z(\mathbf{e}_q)}; J(\mathbf{e}_q)\right)$ fulfills

$$\mathbf{\Delta}_\pi^{-1}\mathbf{E}\left(e^{\beta I(\mathbf{e}_q)}; J(\mathbf{e}_q)\right)^{\mathrm{T}}\mathbf{\Delta}_\pi = \widehat{\mathbf{E}}\left(e^{-\beta Z(\mathbf{e}_q)}; J(\mathbf{e}_q)\right)\;. \qquad (10)$$

Now note that a simple calculation involving Fubini's Theorem shows that

$$\frac{1}{q}\widehat{\mathbb{E}}_{i,0}\left(e^{-\beta Z(\mathbf{e}_q)}\mathbf{1}_{J(\mathbf{e}_q)}\right) = \widehat{\mathbb{E}}_{i,0}\left(\int_0^{\mathbf{e}_q} e^{-\beta Z(u)}\mathbf{1}_{J(u)}du\right).$$

Making use of the Asmussen and Kella's martingale (as given above) applied to the time reversed MAP, it follows by taking expectation with respect to

$\widehat{\mathbb{P}}_{i,0}$ that

$$\widehat{\mathbb{E}}_{i,0}\left(e^{-\beta Z(\mathbf{e}_q)}\mathbf{1}_{J(\mathbf{e}_q)}\right)\left(\widehat{\mathbf{F}}\left(\beta\right)-q\mathbf{I}\right)=-q\mathbf{1}_i+\beta q\widehat{\mathbb{E}}_{i,0}\left(\int_0^{\mathbf{e}_q}\mathbf{1}_{J(u)}dS(u)\right). \tag{11}$$

Now note, with the help of Fubini's Theorem and the fact that $(X,\widehat{\mathbb{P}}_{i,0})$ can only creep upwards thus allowing $\tau^+_\cdot$ to be considered the inverse of $S\left(\cdot\right)$,

$$\begin{aligned}\widehat{\mathbb{E}}_{i,0}\left(\int_0^{\mathbf{e}_q}\mathbf{1}_{J(u)}dS(u)\right)&=\widehat{\mathbb{E}}_{i,0}\left(\int_0^{\infty}e^{-qu}\mathbf{1}_{J(u)}dS(u)\right)\\&=\widehat{\mathbb{E}}_{i,0}\left(\int_0^{\infty}1_{(\tau_a^+<\infty)}e^{-q\tau_a^+}\mathbf{1}_{J(\tau_a^+)}da\right).\end{aligned}$$

Further, applying a change of measure in the spirit of (2) we have

$$\begin{aligned}&\widehat{\mathbb{E}}_{i,0}\left(\int_0^{\mathbf{e}_q}1_{(J(u)=j)}dS(u)\right)\\&=\int_0^{\infty}e^{-\Phi(q)a}\widehat{\mathbb{E}}_{i,0}^{\Phi(q)}\left(1_{(\tau_a^+<\infty)}1_{(J(\tau_a^+)=j)}\right)\frac{\widehat{h}_i(\Phi(q))}{\widehat{h}_j(\Phi(q))}da\,.\end{aligned} \tag{12}$$

(Recall that $\kappa=\widehat{\kappa}$ and hence $\Phi=\widehat{\Phi}$). Finally noting from (3) that $\kappa'_{\Phi(q)}\left(0\right)=\kappa'(\Phi\left(q\right))>0$, it follows that under $\widehat{\mathbb{P}}^{\Phi(q)}_{i,0}$ the process X drifts to infinity thus allowing the removal of the indicator $1_{(\tau_a^+<\infty)}$ in (12). Replacing (10) and (12) in (11) and considering the latter vector equality as the i-th row of a matrix, we obtain

$$\begin{aligned}&\boldsymbol{\Delta}_\pi^{-1}\mathbf{E}\left(e^{\beta I(\mathbf{e}_q)};J(\mathbf{e}_q)\right)^{\mathrm{T}}\boldsymbol{\Delta}_\pi\left(\widehat{\mathbf{F}}\left(\beta\right)-q\mathbf{I}\right)\\&\qquad=-q\mathbf{I}+\beta q\boldsymbol{\Delta}_{\widehat{\mathbf{h}}}(\Phi\left(q\right))\widehat{\mathbf{U}}(q)\boldsymbol{\Delta}_{\widehat{\mathbf{h}}}(\Phi\left(q\right))^{-1},\end{aligned}$$

where

$$\widehat{\mathbf{U}}_{ij}(q)=\int_0^\infty da\cdot e^{-\Phi(q)a}\widehat{\mathbb{P}}^{\Phi(q)}_{i,0}(J(\tau_a^+)=j).$$

Note that matrix $\widehat{\mathbf{U}}(q)$ has no zero columns by the assumption that none of the Lévy processes is downward subordinator. Recalling that $\widehat{\mathbf{F}}\left(\alpha\right)=\boldsymbol{\Delta}_\pi^{-1}\mathbf{F}\left(\alpha\right)^{\mathrm{T}}\boldsymbol{\Delta}_\pi$ we have

$$\begin{aligned}&\mathbf{E}\left(e^{\beta I(\mathbf{e}_q)};J(\mathbf{e}_q)\right)^{\mathrm{T}}\left(\mathbf{F}\left(\beta\right)-q\mathbf{I}\right)^{\mathrm{T}}\\&\qquad=-q\mathbf{I}+\beta q\boldsymbol{\Delta}_\pi\boldsymbol{\Delta}_{\widehat{\mathbf{h}}}(\Phi\left(q\right))\widehat{\mathbf{U}}(q)\boldsymbol{\Delta}_{\widehat{\mathbf{h}}}(\Phi\left(q\right))^{-1}\boldsymbol{\Delta}_\pi^{-1}.\end{aligned}$$

Further noting that $\boldsymbol{\Delta}_\pi\widehat{\mathbf{h}}\left(\alpha\right)^{\mathrm{T}}=\mathbf{v}\left(\alpha\right)^{\mathrm{T}}$ implies that $\boldsymbol{\Delta}_\pi\boldsymbol{\Delta}_{\widehat{\mathbf{h}}}(\alpha)=\boldsymbol{\Delta}_{\mathbf{v}}\left(\alpha\right)$, the diagonal matrix associated with vector $\mathbf{v}\left(\alpha\right)$, we have that

$$\mathbf{E}\left(e^{\beta I(\mathbf{e}_q)}; J(\mathbf{e}_q)\right)^{\mathrm{T}} (\mathbf{F}(\beta) - q\mathbf{I})^{\mathrm{T}} = -q\mathbf{I} + \beta q \boldsymbol{\Delta}_{\mathbf{v}}(\Phi(q))\widehat{\mathbf{U}}(q)\boldsymbol{\Delta}_{\mathbf{v}}(\Phi(q))^{-1}. \tag{13}$$

The theorem is proved once we note that $\widehat{\mathbf{U}}(q)$ is a resolvent and hence equal to $(\Phi(q)\mathbf{I} - \widehat{\boldsymbol{\Lambda}}(q))^{-1}$. □

6 Proof of Theorem 3

(ii) and (iii). We first prove that the Laplace transform in (7) is well-defined, that is $\mathbf{F}(\beta) - q\mathbf{I}$ is invertible for sufficiently large β. From Theorem 4 it follows that $\mathbf{F}(\beta) - q\mathbf{I}$ is invertible for β for which RHS of (9) is invertible. This holds β such that $\beta - \Phi(q)$ is greater than spectrum of matrix $\widehat{\boldsymbol{\Lambda}}(q)$ since:

$$[\beta(\Phi(q)\mathbf{I} - \widehat{\boldsymbol{\Lambda}}(q))^{-1} - \mathbf{I}]^{-1} = -[\mathbf{I} + \beta((\Phi(q) - \beta)\mathbf{I} - \widehat{\boldsymbol{\Lambda}}(q))^{-1}]. \tag{14}$$

Note also that $\mathbf{F}(\beta) - q\mathbf{I}$ is always invertible for $\beta < \Phi(q)$.

According to the definition of $\mathbf{M}^{(q)}$ via its Laplace transform simple manipulation of (9) yields

$$\mathbf{P}_x(I(\mathbf{e}_q) < 0; J(\mathbf{e}_q)) = \mathbf{M}^{(q)}(x)\mathbf{I}(q). \tag{15}$$

Using the law of total probability and first conditioning the probability on the left hand side of (15) with respect to $\mathcal{F}_{\tau_0^-}$ we have

$$\mathbf{E}_x(e^{-q\tau_0^-} 1_{(\tau_0^- < \infty)}; J(\tau_0^-))\mathbf{I}(q) = \mathbf{M}^{(q)}(x)\mathbf{I}(q).$$

and hence the first result follows for $q > 0$.

We may now take limits as $q \downarrow 0$ and note that the limit exists on the right hand side of (8) because it exists on the left hand side. □

(i) and (iv). Assume momentarily that X drifts to infinity. With this assumption, it is clear that X hits every level $a > 0$ with probability one. Next note that for any $0 < x \le a$,

$$\{\mathbf{P}_x(\tau_a^+ < \tau_0^-; J(\tau_a^+))\}_{ij} = \{\mathbb{P}_{i,x}(J(\tau_a^+) = j, \tau_a^+ < \tau_0^-)\}$$

is the transition matrix of the Feller Markov chain $\{J^*(\tau_a^+) : a \ge 0\}$ of the modulating state on the upward ladder-height process of X^*, where X^* is the process X killed on exiting $(0, \infty)$. Note that the transitions of $J^*(\tau_a^+)$ depend on the position of $X^*(\tau_a^+)$ and hence on the level a since on $\tau_a^+ < \tau_0^-$ we have $X^*(\tau_a^+) = a$. Therefore, for each $y > 0$ there exists some $N \times N$ sub-stochastic intensity matrix $\boldsymbol{\Lambda}^*(y)$ such that

$$\mathbf{P}_x(\tau_a^+ < \tau_0^-; J(\tau_a^+)) = \exp\left\{\int_x^a \boldsymbol{\Lambda}^*(y)dy\right\}, \quad x > 0, \tag{16}$$

which has inverse $\exp\{-\int_x^a \mathbf{\Lambda}^*(y)dy\}$. It is also now clear that there exists invertible matrix

$$\mathbf{W}(x) = \exp\left\{-\int_x^b \mathbf{\Lambda}^*(y)dy\right\}$$

for some arbitrary $b > a$ (and therefore determined up to a pre- or post-multiplicative constant, invertible matrix) which is obviously almost everywhere differentiable on $(0,\infty)$. We have thus proved part (ii) of Theorem 3 for the case that $q = 0$ and X drifts to infinity.

Using a change of measure (2) we have for any $q > 0$ that

$$\begin{aligned}
&\mathbb{E}_{i,x}\left(e^{-q\tau_a^+}1_{(\tau_a^+<\tau_0^-,J(\tau_a^+)=j)}\right)\\
&\quad= \mathbb{E}_{i,x}\left(e^{\Phi(q)(a-x)-q\tau_a^+}1_{(\tau_a^+<\tau_0^-,J(\tau_a^+)=j)}\frac{h_j(\Phi(q))}{h_i(\Phi(q))}\right)\frac{h_i(\Phi(q))}{h_j(\Phi(q))}e^{-\Phi(q)(a-x)}\\
&\quad= h_i(\Phi(q))\mathbb{P}_{i,x}^{\Phi(q)}\left(\tau_a^+ < \tau_0^-, J(\tau_a^+) = j\right)\frac{1}{h_j(\Phi(q))}e^{-\Phi(q)(a-x)}. \qquad (17)
\end{aligned}$$

Now $\kappa'_{\Phi(q)}(0) = \kappa'(\Phi(q)) > 0$. So that under $\mathbb{P}_{i,x}^{\Phi(q)}$ the process X always drifts to infinity. We may thus use the conclusions of the previous paragraph in (17) to deduce

$$\begin{aligned}
&\mathbf{E}_x\left(e^{-q\tau_a^+}1_{(\tau_a^+<\tau_0^-)}; J(\tau_a^+)\right)\\
&\quad= [\mathbf{\Delta_h}(\Phi(q))\, e^{\Phi(q)x}\mathbf{W}_{\Phi(q)}(x)\,\mathbf{\Delta_h}(\Phi(q))^{-1}]\\
&\qquad\times[\mathbf{\Delta_h}(\Phi(q))\, e^{\Phi(q)a}\mathbf{W}_{\Phi(q)}(a)\,\mathbf{\Delta_h}(\Phi(q))^{-1}]^{-1}\\
&\quad= \mathbf{W}^{(q)}(x)\,\mathbf{W}^{(q)}(a)^{-1} \qquad (18)
\end{aligned}$$

where in the first equality we understand $\mathbf{W}_{\Phi(q)}(x)$ to be the scale function associated with $(X, P^{\Phi(q)})$ and the equality itself follows from the analysis in the case that X drifts to infinity above. Further, in the last line of (18) uses the definition

$$\mathbf{W}^{(q)}(x) = \mathbf{\Delta_h}(\Phi(q))\, e^{\Phi(q)x}\mathbf{W}_{\Phi(q)}(x)\,\mathbf{\Delta_h}(\Phi(q))^{-1}. \qquad (19)$$

Note that $\mathbf{W}^{(q)}(x)$ is almost everywhere differentiable since $\mathbf{W}_{\Phi(q)}$ is. □

(v). This result follows from elementary linear algebra. Simply note from the Strong Markov Property that for $q > 0$,

$$\begin{aligned}
&\mathbf{E}_x(e^{-q\tau_0^-}1_{(\tau_0^-<\infty)}; J(\tau_0^-))\\
&\quad= \mathbf{E}_x(e^{-q\tau_0^-}1_{(\tau_0^-<\tau_a^+)}; J(\tau_0^-))\\
&\qquad+\mathbf{E}_x(e^{-q\tau_0^-}1_{(\tau_0^->\tau_a^+)}; J(\tau_0^-))\\
&\quad= \mathbf{E}_x(e^{-q\tau_0^-}1_{(\tau_0^-<\tau_a^+)}; J(\tau_0^-))\\
&\qquad+\mathbf{E}_x(e^{-q\tau_a^+}1_{(\tau_0^->\tau_a^+)}; J(\tau_a^+))\mathbf{E}_a(e^{-q\tau_0^-}1_{(\tau_0^-<\infty)}; J(\tau_0^-)).
\end{aligned}$$

Rearranging and substituting in the conclusions of part (i) and (ii) gives the required result for $q > 0$. To deal with the case that $q = 0$, take the limit above as q tends to zero. □

7 An example

In general, it is difficult to identify the scale matrix $\mathbf{W}^{(q)}(x)$. Note that using the Strong Markov Property we have for $0 < x \leq a$,

$$\begin{aligned}
&\mathbf{P}_x(\tau_a^+ < \tau_0^-; J(\tau_a^+)) = \mathbf{P}_x(J(\tau_a^+)) - \mathbf{P}_x(\tau_0^- < \tau_a^+; J(\tau_a^+)) \\
&\quad = \mathbf{P}_x(J(\tau_a^+)) - \int_{-\infty}^0 \mathbf{P}_x(\tau_0^- < \tau_a^+; X(\tau_0^-) \in dy; J(\tau_0^-))\mathbf{P}_y(J(\tau_a^+)) \\
&\quad = e^{\mathbf{\Lambda}(0)(a-x)} - \int_{-\infty}^0 \mathbf{P}_x(\tau_0^- < \tau_a^+; X(\tau_0^-) \in dy; J(\tau_0^-))e^{\mathbf{\Lambda}(0)(a-y)} \\
&\quad = \mathbf{W}(x)\mathbf{W}(a)^{-1},
\end{aligned}$$

where $\mathbf{W}(x) = \mathbf{W}^{(0)}(x)$. Thus

$$\begin{aligned}
&\mathbf{W}(x)[e^{\mathbf{\Lambda}(0)a}\mathbf{W}(a)]^{-1} \\
&\quad = e^{-\mathbf{\Lambda}(0)x} - \int_{-\infty}^0 \mathbf{P}_x(\tau_0^- < \tau_a^+; X(\tau_0^-) \in dy; J(\tau_0^-))e^{-\mathbf{\Lambda}(0)y}. \quad (20)
\end{aligned}$$

Note the intensity matrix $\mathbf{\Lambda}(0)$ has eigenvalues with non-positive real part and hence the integral on the right hand side of the above identity makes sense. Further, it has a limit as $a \to \infty$. This implies that the left hand side of (20) also has a limit. Thus up to some invertible constant matrix $\mathbf{V}$ we have

$$\mathbf{W}(x) = \mathbf{V}\left[e^{-\mathbf{\Lambda}(0)x} - \int_{-\infty}^0 \mathbf{P}_x(\tau_0^- < \infty; X(\tau_0^-) \in dy; J(\tau_0^-))e^{-\mathbf{\Lambda}(0)y}\right]. \quad (21)$$

Without loss of generality we can assume further that $\mathbf{V} = \mathbf{I}$. Note also that

$$\begin{aligned}
&\mathbf{E}_x\left[e^{-qX(\tau_0^-)}1_{(\tau_0^-<\infty)}; J(\tau_0^-)\right] \\
&\quad = \mathbf{\Delta_h}(q)\,\mathbf{E}_x^q\left[e^{\kappa(q)\tau_0^-}1_{(\tau_0^-<\infty)}; J(\tau_0^-)\right]\mathbf{\Delta_h}(q)^{-1}, \quad (22)
\end{aligned}$$

and the latter can be derived from (8).

Here is an example of $\mathbf{W}^{(q)}(x)$ that can be identified. We consider *Brownian motion* and independent *Markov chain.* In other words $X^{(1)}(t) = 0$ for all $t \geq 0$ and $\psi_i(\beta) = \beta^2/2$ for each $i = 1, ..., N$. For this case, because the trajectories of the Brownian motion are continuous,

$$\begin{aligned}
\mathbf{W}(x) &= e^{-\mathbf{\Lambda}(0)x} - \mathbf{P}_x(\tau_0^- < \infty; J(\tau_0^-)) \\
&= e^{-\mathbf{\Lambda}(0)x} - \mathbf{M}^{(0)}(x).
\end{aligned}$$

Denote by λ_i the eigenvalues of $\mathbf{Q}$ (which have non-positive real part by Theorem 2.5 of Seneta (1973)) and denote by $\mathbf{H}$ the matrix of eigenvectors of $\mathbf{Q}$. Hence $\mathbf{Q} = \mathbf{H}^{-1}\mathrm{diag}(\lambda_i)\mathbf{H}$. Then $\mathbf{F}(\beta) = \mathbf{H}^{-1}\mathrm{diag}(\beta^2/2 + \lambda_i)\mathbf{H}$ and thus $\mathbf{\Lambda}(0) = \mathbf{H}^{-1}\mathrm{diag}((-2\lambda_i)^{1/2})\mathbf{H}$. Straightforward calculations based on the definition of matrix $\mathbf{M}^{(0)}(x)$ gives that $\int_0^\infty e^{-\beta x}\mathbf{W}(x)\,dx = \mathbf{C}\mathbf{F}(\beta)^{-1}$, where $\mathbf{C} = \mathbf{H}^{-1}\mathrm{diag}((-2\lambda_i/2)^{1/2})\mathbf{H}$. From the definition of the matrix $\mathbf{W}^{(q)}(x)$ given in (19) we have

$$\begin{aligned}\int_0^\infty e^{-\beta x}\mathbf{W}^{(q)}(x)\,dx &= \varDelta_{\mathbf{h}}(\varPhi(q))\int_0^\infty e^{-(\beta-\varPhi(q))x}\mathbf{W}_{\varPhi(q)}(x)\,dx\varDelta_{\mathbf{h}}(\varPhi(q))^{-1}\\ &= \varDelta_{\mathbf{h}}(\varPhi(q))\,\mathbf{C}_{\varPhi(q)}\varDelta_{\mathbf{h}}(\varPhi(q))^{-1}\cdot\\ &\quad\times\varDelta_{\mathbf{h}}(\varPhi(q))\,\mathbf{F}_{\varPhi(q)}(\beta-\varPhi(q))^{-1}\varDelta_{\mathbf{h}}(\varPhi(q))^{-1},\end{aligned}$$

where $\mathbf{C}_{\varPhi(q)}$ plays the role of the matrix $\mathbf{C}$ for $(X, \mathbf{P}^{\varPhi(q)})$. From (3) we can check that

$$\varDelta_{\mathbf{h}}(\varPhi(q))\,\mathbf{F}_{\varPhi(q)}(\beta-\varPhi(q))^{-1}\varDelta_{\mathbf{h}}(\varPhi(q))^{-1} = (\mathbf{F}(\beta) - q\mathbf{I})^{-1}.$$

Thus defining $\mathbf{C}(q) = \varDelta_{\mathbf{h}}(\varPhi(q))\,\mathbf{C}_{\varPhi(q)}\varDelta_{\mathbf{h}}(\varPhi(q))^{-1}$ we deduce that

$$\int_0^\infty e^{-\beta x}\mathbf{W}^{(q)}(x)\,dx = \mathbf{C}(q)\times(\mathbf{F}(\beta) - q\mathbf{I})^{-1}.$$

From above we derive $\mathbf{W}^{(q)}(x) = \mathbf{H}^{-1}\mathrm{diag}(\sinh(x(2q-2\lambda_i)^{1/2}))\mathbf{H}$ up to multiplicative constant invertible matrix. Thus from Theorem 3 we obtain e.g. the state of the independent Markov chain at the exit time of the Brownian motion from the interval:

$$\mathbf{E}_x\left(e^{-q\tau_a^+}1_{(\tau_a^+<\tau_0^-)}; J(\tau_a^+)\right) = \mathbf{H}^{-1}\mathrm{diag}\left(\frac{\sinh(x(2q-2\lambda_i)^{1/2})}{\sinh(a(2q-2\lambda_i)^{1/2})}\right)\mathbf{H}.$$

Remark 5. For $N = 1$, that is for spectrally negative Lévy process with Laplace exponent $\psi(\beta) = F(\beta)$, from (21) we have $W^{(0)}(x) = V(1 - M^{(0)}(x))$ for some constant $V > 0$ and hence from Theorem 3 and (19) we derive (up to a multiplicative constant) $\int_0^\infty e^{-\beta x}W^{(q)}(x)\,dx = (\psi(\beta) - q)^{-1}$ (compare with Theorem 1 of Kyprianou and Palmowski (2004)).

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References

1. Asmussen, S. (1989) Aspects of matrix Wiener-Hopf factorization in applied probability. *Math. Scientist.* **14**, 101–116.
2. Asmussen, S. (2000) *Ruin Probabilities.* World Scientific, Singapore.
3. Asmussen, S. (2003) *Applied Probability and Queues* Second edition. Springer.
4. Asmussen, A., Avram, F. and Pistorius, M. R. (2004) Russian and American put options under exponential phase-type Lévy models. *Stoch. Proc. Appl.* **109**, 79–111.
5. Asmussen, S. and Kella, O. (2000) Multi-dimensional martingale for Markov additive processes and its applications. *Adv. Appl. Probab.* **32(2)**, 376–380.
6. Avram, F., Kyprianou, A.E. and Pistorius, M.R. (2004) Exit problems for spectrally negative Lévy processes and applications to (Canadized) Russian options. *Ann. Appl. Probab.* **14**, 215–238.
7. Bertoin, J. (1997) Exponential decay and ergodicity of completely asymmetric Lévy processes in a finite interval, *Ann. Appl. Probab.* **7**, 156–169.
8. Bingham, N.H. (1975) Fluctuation theory in continuous time. *Adv. Appl. Probab.* **7**, 705–766.
9. Chan, T. (1998) Some applications of Lévy processes to stochastic investment models for actuarial use. *ASTIN Bulletin* **28**, 77–93.
10. Chiu, S.N. and Yin, C. (2005) Passage times for a spectrally negative Lévy process with applications to risk theory. *Bernoulli* **11(3)**, 511–522.
11. Coolen-Schrijner, P. and van Doorn, E.A. (2002) The deviation matrix of a continuous-time Markov chain. *Probab. Engrg. Inform. Sci.* **16(3)**, 351–366.
12. Dube, P., Guillemin, F. and Mazumdar, R. (2004) Scale functions of Lévy processes and busy periods of finite capacity $M/GI/1$ queues. *J. Appl. Probab.* **41(4)**, 1250-1254.
13. Emery, D.J. (1973) Exit problems for a spectrally positive process. *Adv. Appl. Prob.* **5**, 498–520.
14. Huzak, M., Perman, M., Šikić, H. and Vondraček, Z. (2004), Ruin probabilities and decompositions for general perturbed risk processes. *Ann. Appl. Probab.* **14**, 1378–1397.
15. Jacod, J. and Shiryaev, A.N. (1987) *Limit Theorems for Stochastic Processes.* Springer.
16. Kella, O. and Whitt, W. (1992) Useful martingales for stochastic storage processes with Lévy input. *J. Appl. Probab.* **29**, 396–403.
17. Klüppelberg, C., Kyprianou, A.E. and Maller, R.A. (2003) Ruin probabilties and overshoots for general Lévy insurance risk process. *Ann. Appl. Probab.* **14(4)**, 1766–1801.
18. Kyprianou, A. E. and Palmowski, Z. (2004) A martingale review of some fluctuation theory for spectrally negative Lévy processes. *Séminaire de Probabilités XXXVIII*, 16–29.
19. Miyazawa, M. (2004) Hitting probabilities in a Markov Additive process with linear movements and upward jumps: applications to risk and queueing processes. *Ann. Appl. Probab.* **14(2)**, 1029–1054.
20. Miyazawa, M. and Takada, H. (2002) A matrix exponential form for hitting probabilities and its application to a Markov modulated fluid queue with downward jumps. *J. Appl. Probab.* **39(3)**, 604-618.
21. Palmowski, Z. and Rolski, T. A technique for exponential change of measure for Markov processes. *Bernoulli* **8**, 767-785

22. Pistorius, M.R. (2003) *Exit problems for Lévy processes and applications to mathematical finance.* Ph.D Thesis, University of Utrecht.
23. Pistorius, M.R. (2006) On maxima and ladder processes for a dense class of Lévy processes. To appear in *Journal of Applied Probability* **43**.
24. Rogers, L.C.G. (1990) The two-sided exit problem for spectrally positive Lévy processes. *Adv. Appl. Probab.* **22**, 486–487.
25. Rogers, L.C.G. (1994) Fluid models in queueing theory and Wiener-Hopf factorization of Markov chains. *Ann. Appl. Probab.* **4(2)**, 390–413.
26. Schmidli, H. (1999) Perturbed risk processes: a review. *Theor. Stoch. Proc.* **5**, 145–165.
27. Seneta, E. (1973) *Non-negative Matrices. An Introduction to Theory and Applications.* George Allen & Unwin Ltd, London.
28. Suprun, V.N. (1976), Problem of destruction and resolvent of terminating processes with independent increments. *Ukranian Math. J.* **28**, 39–45.
29. Takács, L. (1967) *Combinatorial methods in the theory of stochastic processes.* John Wiley & Sons, Inc.
30. Zolotarev, V.M. (1964) The first passage time of a level and the behavior at infinity for a class of processes with independent increments. *Theory Prob. Appl.* **9**, 653–661.